Representations in symbolic form

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Some notes on linear maps, changes of bases, and linear systems follow, that try to clarify the relations between "abstract" vector spaces and representations using an "agile" notation in order to avoid relying on summations. The following is not strictly rigorous (especially regarding derivatives), these notes are meant only to support intuition.

Representation of a vector

Let V be an n-dimensional vector space over $\mathbb{R}$, and $\{\mathbf{v}_1 \cdots \mathbf{v}_n\}$ be a base. As we know, every vector $\mathbf{v} \in V$ admits an unique representation

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{v}_i \; v_i$$

where $v_1 \cdots v_n \in \mathbb{R}$ are called "coordinates". To simplify formulas, in this section we will use systematically the following notation:

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{v}_i \; v_i = \{\mathbf{v}_1 \cdots \mathbf{v}_n\} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

The right-hand side must be interpreted as a "symbolic dot product", where the row is indeed a "row of vectors".

Representation of a linear map

Let $\{\mathbf{v}_1 \cdots \mathbf{v}_n\}$ be a base of the vector space V, $\{\mathbf{w}_1 \cdots \mathbf{w}_m\}$ a base of the m-dimensional vector space W, and $\mathcal{A} : V \to W$ a linear map. The behavior of $\mathcal{A}$ is captured by its action on any base of V. Let us write it down:

$$\mathcal{A}(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m = \{\mathbf{w}_1 \cdots \mathbf{w}_m\} \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\mathcal{A}(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m = \{\mathbf{w}_1 \cdots \mathbf{w}_m\} \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vdots$$

$$\mathcal{A}(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m = \{\mathbf{w}_1 \cdots \mathbf{w}_m\} \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Then, stacking the columns and post-multiplying by $\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$, we obtain

$$\{\mathcal{A}(\mathbf{v}_1)\cdots\mathcal{A}(\mathbf{v}_n)\}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \{\mathbf{w}_1\cdots\mathbf{w}_m\}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Now we manipulate a bit the left-hand side:

$$\{\mathcal{A}(\mathbf{v}_1)\cdots\mathcal{A}(\mathbf{v}_n)\}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n}\mathcal{A}(\mathbf{v}_i)v_i = \mathcal{A}\left(\sum_{i=1}^{n}\mathbf{v}_i\ v_i\right) = \mathcal{A}\left(\{\mathbf{v}_1\cdots\mathbf{v}_n\}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)$$

And finally we get, for a generic vector $\mathbf{v}$:

$$\begin{aligned}\mathcal{A}(\mathbf{v}) = \mathcal{A}\left(\{\mathbf{v}_1\cdots\mathbf{v}_n\}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) &= \{\mathbf{w}_1\cdots\mathbf{w}_m\}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \{\mathbf{w}_1\cdots\mathbf{w}_m\}\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \mathbf{w}\end{aligned}$$

This tells us that, once the bases $\{\mathbf{v}_1\cdots\mathbf{v}_n\}$ and $\{\mathbf{w}_1\cdots\mathbf{w}_m\}$ are fixed, the relation $\mathcal{A}(\mathbf{v}) = \mathbf{w}$ can be represented uniquely by a relation on the respective coordinates:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

or, which is the same, that every linear map $\mathcal{A}$ can be represented uniquely by a matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$.

Composition of linear maps

Let us see how the composition of linear maps reflects in the multiplications of matrices (exercise 1 in ex. paper 2). Let U, V, W be vector spaces with respective bases $\{\mathbf{u}_1\cdots\mathbf{u}_m\}$, $\{\mathbf{v}_1\cdots\mathbf{v}_n\}$, $\{\mathbf{w}_1\cdots\mathbf{w}_m\}$, and let $\mathcal{A} : V \to W$, $\mathcal{B} : U \to V$ be represented, respectively, by the matrices A and B. This means:

$$\begin{aligned}\mathcal{A}(\mathbf{v}) &= \mathcal{A}\left(\{\mathbf{v}_1\cdots\mathbf{v}_n\}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = \{\mathbf{w}_1\cdots\mathbf{w}_m\}A\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{w} \\ \mathcal{B}(\mathbf{u}) &= \mathcal{B}\left(\{\mathbf{u}_1\cdots\mathbf{u}_m\}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\right) = \{\mathbf{v}_1\cdots\mathbf{v}_n\}B\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \mathbf{v}\end{aligned}$$

But then,

$$\mathcal{A}\circ\mathcal{B}(\mathbf{u}) = \mathcal{A}\circ\mathcal{B}\left(\{\mathbf{u}_1\cdots\mathbf{u}_m\}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\right) = \mathcal{A}\left(\mathcal{B}\left(\{\mathbf{u}_1\cdots\mathbf{u}_m\}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\right)\right)$$
$$= \mathcal{A}\left(\{\mathbf{v}_1\cdots\mathbf{v}_n\}B\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\right) = \{\mathbf{w}_1\cdots\mathbf{w}_m\}A\ B\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \mathbf{w}$$

Thus, the relation $\mathcal{A}\circ\mathcal{B}(\mathbf{u}) = \mathbf{w}$ is represented by the relation on the respective coordinates:

$$A\ B\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

or, which is the same, $\mathcal{A}\circ\mathcal{B}$ is represented by AB.

Change of base

Let X be a vector space, and $\{\mathbf{x}_1\cdots\mathbf{x}_n\}$, $\{\tilde{\mathbf{x}}_1\cdots\tilde{\mathbf{x}}_n\}$ be two bases. Hence, any vector $\mathbf{x}\in X$ admits the distinct representations:

$$\mathbf{x} = \{\mathbf{x}_1\cdots\mathbf{x}_n\}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \{\tilde{\mathbf{x}}_1\cdots\tilde{\mathbf{x}}_n\}\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$$

How are these related? If we are given, say, coordinates $\tilde{x}_1\cdots\tilde{x}_n$ with respect to the second base, how can we obtain the corresponding coordinates with respect to the first? Since $\{\mathbf{x}_1\cdots\mathbf{x}_n\}$ is a base, in particular every vector of $\{\tilde{\mathbf{x}}_1\cdots\tilde{\mathbf{x}}_n\}$ admits a representation with respect to it:

$$\tilde{\mathbf{x}}_1 = t_{11}\mathbf{x}_1 + t_{21}\mathbf{x}_2 + \cdots + t_{n1}\mathbf{x}_n = \{\mathbf{x}_1\cdots\mathbf{x}_n\}\begin{bmatrix} t_{11} \\ \vdots \\ t_{n1} \end{bmatrix}$$
$$\vdots$$
$$\tilde{\mathbf{x}}_n = t_{1n}\mathbf{x}_1 + t_{2n}\mathbf{x}_2 + \cdots + t_{nn}\mathbf{x}_n = \{\mathbf{x}_1\cdots\mathbf{x}_n\}\begin{bmatrix} t_{1n} \\ \vdots \\ t_{nn} \end{bmatrix}$$

Stacking the columns and post-multiplying by $\begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_n \end{bmatrix}^T$, we obtain

$$\mathbf{x} = \{\tilde{\mathbf{x}}_1\cdots\tilde{\mathbf{x}}_n\}\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \{\mathbf{x}_1\cdots\mathbf{x}_n\}\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \{\mathbf{x}_1\cdots\mathbf{x}_n\}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Thus, in a *change of base* from $\{\tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n\}$ to $\{\mathbf{x}_1 \cdots \mathbf{x}_n\}$, the coordinates of a vector $\mathbf{x}$ change according to a premultiplication by a matrix T:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = T \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$$

(the coordinates change, but the vector $\mathbf{x}$ remains the same!). It is also said that T "represents" the change of base. Note that, in view of the previous section, T actually represents *the identity map* $\mathrm{Id} : X \to X$, where the "first copy" of X (the domain) has the base $\{\tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n\}$, whereas the second (the codomain) has the base $\{\mathbf{x}_1 \cdots \mathbf{x}_n\}$.

Changes of base and linear maps

Let as before V and W be vector spaces, $\{\mathbf{v}_1 \cdots \mathbf{v}_n\}$, $\{\mathbf{w}_1 \cdots \mathbf{w}_m\}$ respective bases, and $\mathcal{A} : V \to W$ a linear map. As we know, $\mathcal{A}$ is represented by a matrix A which is determined uniquely once the bases are fixed:

$$\mathcal{A}\left(\{\mathbf{v}_1 \cdots \mathbf{v}_n\} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) = \{\mathbf{w}_1 \cdots \mathbf{w}_m\} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \quad \leftrightarrow \quad A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

But suppose that we *change* both the bases to new bases $\{\tilde{\mathbf{v}}_1 \cdots \tilde{\mathbf{v}}_n\}$, $\{\tilde{\mathbf{w}}_1 \cdots \tilde{\mathbf{w}}_n\}$. How does A change accordingly? As we saw in the previous section, there exist two matrices T_V and T_W that represent, respectively, the change of base from $\{\tilde{\mathbf{v}}_1 \cdots \tilde{\mathbf{v}}_n\}$ to $\{\mathbf{v}_1 \cdots \mathbf{v}_n\}$ and from $\{\tilde{\mathbf{w}}_1 \cdots \tilde{\mathbf{w}}_n\}$ to $\{\mathbf{w}_1 \cdots \mathbf{w}_m\}$:

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = T_V \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix} \qquad \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = T_W \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{bmatrix}$$

Thus, the relation $\mathcal{A}(\mathbf{v}) = \mathbf{w}$ reads (remember that each vector into play remains the same, only representations change):

$$\begin{aligned} \mathcal{A}(\mathbf{v}) &= \mathcal{A}\left(\{\tilde{\mathbf{v}}_1 \cdots \tilde{\mathbf{v}}_n\} \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix} \right) = \mathcal{A}\left(\{\mathbf{v}_1 \cdots \mathbf{v}_n\} T_V \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix} \right) \\ &= \mathbf{w} = \{\tilde{\mathbf{w}}_1 \cdots \tilde{\mathbf{w}}_n\} \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{bmatrix} = \{\mathbf{w}_1 \cdots \mathbf{w}_m\} T_W \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{bmatrix} \end{aligned}$$

Comparing the rightmost expressions and remembering the meaning of A, we finally obtain:

$$A\ T_V \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix} = T_W \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{bmatrix}$$

or

$$T_W^{-1}\ A\ T_V \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix} = \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{bmatrix}$$

Thus, $T_W^{-1}\ A\ T_V$ is the representation of $\mathcal{A}$ with respect to the new bases.

Functions of time and derivatives

We use functions like $\mathbf{x} : \mathbb{R}_+ \to X$, that at each "time" t associates a state $\mathbf{x}(t)$, to represent curves, trajectories in the state space, solutions to differential equations etc. Let us fix a base $\{\mathbf{x}_1 \cdots \mathbf{x}_n\}$ of the state space. Then, as happens to every vector in X, to each value assumed by the function there corresponds a set of coordinates:

$$\mathbf{x}(t) = \{\mathbf{x}_1 \cdots \mathbf{x}_n\} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

In particular, we may view the coordinates as functions of time by themselves, or the column vector $\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$ as a function of time. If the function $\mathbf{x} : \mathbb{R}_+ \to X$ is sufficiently regular, we may form its derivative at t:

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \lim_{h\to 0} \frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h}$$

But now exploiting our symbolic notation:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \lim_{h\to 0} \frac{\{\mathbf{x}_1 \cdots \mathbf{x}_n\} \begin{bmatrix} x_1(t+h) \\ \vdots \\ x_n(t+h) \end{bmatrix} - \{\mathbf{x}_1 \cdots \mathbf{x}_n\} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}{h} \\
&= \{\mathbf{x}_1 \cdots \mathbf{x}_n\} \lim_{h\to 0} \frac{\begin{bmatrix} x_1(t+h) \\ \vdots \\ x_n(t+h) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}{h} \\
&= \{\mathbf{x}_1 \cdots \mathbf{x}_n\} \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}
\end{aligned}$$

In words, if the base is fixed and constant, *the derivatives of the coordinates are the coordinates of the derivative.*

Representation of a linear system

Now it should be clear how to deal with a linear system. Let U, Y, X be vector spaces with respective fixed bases $\{\mathbf{u}_1 \cdots \mathbf{u}_m\}, \{\mathbf{y}_1 \cdots \mathbf{y}_p\}, \{\mathbf{x}_1 \cdots \mathbf{x}_n\}$. A linear system is a differential equation in this form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathcal{A}(\mathbf{x}(t)) + \mathcal{B}(\mathbf{u}(t))\\ \mathbf{y}(t) &= \mathcal{C}(\mathbf{x}(t)) + \mathcal{D}(\mathbf{u}(t))\end{aligned}$$

where $\mathcal{A} : X \to X$, $\mathcal{B} : U \to X$, $\mathcal{C} : X \to Y$, $\mathcal{D} : U \to Y$ are linear maps. So far, everything is in an "abstract" setting. Only since the bases have been fixed we can make them explicit:

$$\begin{aligned}\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}\dot{x}_1(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix} &= \mathcal{A}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{B}\left(\{\mathbf{u}_1 \cdots \mathbf{u}_m\}\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}\right)\\ \{\mathbf{y}_1 \cdots \mathbf{y}_p\}\begin{bmatrix}y_1(t)\\ \vdots\\ y_p(t)\end{bmatrix} &= \mathcal{C}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{D}\left(\{\mathbf{u}_1 \cdots \mathbf{u}_m\}\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}\right)\end{aligned}$$

and write everything in terms of the respective representations:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix} &= A\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix} + B\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}\\ \begin{bmatrix}y_1(t)\\ \vdots\\ y_p(t)\end{bmatrix} &= C\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix} + D\begin{bmatrix}u_1(t)\\ \vdots\\ u_m(t)\end{bmatrix}\end{aligned}$$

With respect to such representations, changes of base apply like we have seen before. For example, if $\{\tilde{\mathbf{u}}_1 \cdots \tilde{\mathbf{u}}_n\}$ is another base of U, we know that the change of base from $\{\tilde{\mathbf{u}}_1 \cdots \tilde{\mathbf{u}}_n\}$ to $\{\mathbf{u}_1 \cdots \mathbf{u}_m\}$ is represented by a matrix T_U. Substituting accordingly into the equations of the system we obtain:

$$\begin{aligned}\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}\dot{x}_1(t)\\ \vdots\\ \dot{x}_n(t)\end{bmatrix} &= \mathcal{A}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{B}\left(\{\tilde{\mathbf{u}}_1 \cdots \tilde{\mathbf{u}}_n\}\begin{bmatrix}\tilde{u}_1(t)\\ \vdots\\ \tilde{u}_n(t)\end{bmatrix}\right)\\ &= \mathcal{A}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{B}\left(\{\mathbf{u}_1 \cdots \mathbf{u}_m\}T_U\begin{bmatrix}\tilde{u}_1(t)\\ \vdots\\ \tilde{u}_n(t)\end{bmatrix}\right)\\ \{\mathbf{y}_1 \cdots \mathbf{y}_p\}\begin{bmatrix}y_1(t)\\ \vdots\\ y_p(t)\end{bmatrix} &= \mathcal{C}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{D}\left(\{\tilde{\mathbf{u}}_1 \cdots \tilde{\mathbf{u}}_n\}\begin{bmatrix}\tilde{u}_1(t)\\ \vdots\\ \tilde{u}_n(t)\end{bmatrix}\right)\\ &= \mathcal{C}\left(\{\mathbf{x}_1 \cdots \mathbf{x}_n\}\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right) + \mathcal{D}\left(\{\mathbf{u}_1 \cdots \mathbf{u}_m\}T_U\begin{bmatrix}\tilde{u}_1(t)\\ \vdots\\ \tilde{u}_n(t)\end{bmatrix}\right)\end{aligned}$$

which, remembering the meaning of B and D, is represented as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + B\ T_U \begin{bmatrix} \tilde{u}_1(t) \\ \vdots \\ \tilde{u}_n(t) \end{bmatrix}$$
$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} = C \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + D\ T_U \begin{bmatrix} \tilde{u}_1(t) \\ \vdots \\ \tilde{u}_n(t) \end{bmatrix}$$

hence (A, BT_U, C, DT_U) is the new representation.
This said, we usually deal with linear systems written in the following notation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. What does it mean? One possibility is of course that this is a representation, i.e. that $x(t)$, $u(t)$ and $y(t)$ are the columns of coordinates of some vectors $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{y}(t)$ belonging to "abstract" vector spaces. In this case the above notes apply without further observations.
The most frequent case, however, is that $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$ are regarded as the "abstract" spaces themselves. This may lead to some confusion regarding what a "change of base" may be. Indeed, to help intuition, you should *always* reason in terms of representations. You can *always* assume that those "columns of numbers" $x(t)$, $u(t)$, $y(t)$ are also "columns of coordinates" with respect to some base, *and if such base is not mentioned at all, you can assume without problems that it is the canonical one*, i.e. $\{\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^T, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T\}$.