Notes on Lyapunov's theorem

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The following notes contain the proof of Lyapunov's theorem for stability and asymptotic stability of an equilibrium point of a nonlinear system, along with applications to the proof of asymptotic stability of an equilibrium point via linearization, plus some comments on unstable equilibrium points. The material is adapted from FORNASINI & MARCHESINI, *Appunti di teoria dei sistemi* (in Italian); the interested reader can find a general and broad exposition of Lyapunov theory in KHALIL, *Nonlinear systems*.

1 Preliminaries

We will deal with a continuous time, autonomous, time-invariant nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

For simplicity, suppose that $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and f is continuous. Suppose also that (1) has an equilibrium point at the origin (this is again for simplicity, all the results hold in general), i.e. it holds

$$f(0) = 0$$

Let $\varphi(t; 0, \bar{x})$ denote the unique solution $x(t)$ to (1) that corresponds to $x(0) = \bar{x}$.

Stability

In the following, $\mathcal{B}(\bar{x}, \varepsilon)$ will denote the *open* ball centered at $\bar{x}$ of radius ε, that is the set $\{x \in \mathbb{R}^n : \|x - \bar{x}\| < \varepsilon\}$; $\bar{\mathcal{B}}(\bar{x}, \varepsilon)$ will denote the *closed* ball, or the set $\{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq \varepsilon\}$; and $\mathcal{S}(\bar{x}, \varepsilon)$ will denote the *sphere*, or the set $\{x \in \mathbb{R}^n : \|x - \bar{x}\| = \varepsilon\}$. Note that $\mathcal{S}(\bar{x}, \varepsilon)$ is the boundary of both $\mathcal{B}(\bar{x}, \varepsilon)$ and $\bar{\mathcal{B}}(\bar{x}, \varepsilon)$.

The following definitions of stability are due to Aleksandr Lyapunov (1857-1918). An equilibrium point x_e of a nonlinear system is said to be *stable*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\bar{x} \in \mathcal{B}(x_e, \delta) \Rightarrow \varphi(t; 0, \bar{x}) \in \mathcal{B}(x_e, \varepsilon) \text{ for all } t \geq 0$$

(In essence, the Lyapunov stability of x_e asserts a "simultaneous continuity" — more precisely the *equicontinuity* — at x_e of all the functions in the family $\{\Phi_t : \bar{x} \mapsto \varphi(t; 0, \bar{x})\}_{t \geq 0}$.)

The equilibrium point x_e is said to be *asymptotically stable*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\bar{x} \in \mathcal{B}(x_e, \delta) \Rightarrow \begin{cases} \varphi(t; 0, \bar{x}) \in \mathcal{B}(x_e, \varepsilon) \text{ for all } t \geq 0 \\ \lim_{t \to +\infty} \varphi(t; 0, \bar{x}) = x_e \end{cases}$$

It is clear from the definitions that asymptotic stability implies stability.

Lyapunov functions

Here, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if $V(0) = 0$ and there exists an open ball $B = \mathcal{B}(0, \varepsilon)$ such that $V(x) > 0$ for all $x \in B$. The function V is called *positive semi-definite* if there exists a B such that $V(0) = 0$ and $V(x) \geq 0$ for all $x \in B$.
Analogously, V is called *negative definite* or *negative semi-definite* if $V(0) = 0$ and, respectively, $V(x) < 0$ for all $x \in B, x \neq 0$ or $V(x) \leq 0$ for all $x \in B$.
Positive definite (semi-definite, etc.) quadratic forms, i.e. functions of the form $V(x) = x^\top P x$ where $P \in \mathbb{R}^{n\times n}$, $P = P^\top > 0$ (≥ 0, etc.) are positive definite (semi-definite, etc.) with respect to the above definition. In the literature on system theory positive definite functions, when used to establish stability, are called *Lyapunov functions*. Intuitively, their physical meaning is that of an "energy".

Given the *gradient* of V,

$$\nabla V(x) := \begin{bmatrix} \frac{\partial V(x)}{\partial x_1} & \frac{\partial V(x)}{\partial x_2} & \cdots & \frac{\partial V(x)}{\partial x_n} \end{bmatrix}$$

consider the function $\dot{V} : \mathbb{R}^n \to \mathbb{R}$, called the *Lie derivative* of V and defined as follows:

$$\dot{V}(x) = \text{``}\ \nabla V(x) \cdot \dot{x}\ \text{''} = \nabla V(x) \cdot f(x) = \frac{\partial V(x)}{\partial x_1} f_1(x) + \cdots + \frac{\partial V(x)}{\partial x_n} f_n(x)$$

$\dot{V}$ is a function *of the state*. $\dot{V}(\bar{x})$, evaluated at a certain state $\bar{x}$, gives the rate of increase of V, that is the derivative of V with respect to *time*, along the trajectory of the system passing through $\bar{x}$. The essential fact about Lyapunov functions is that the computation of such rate does not require the preliminary computation of the trajectory.

2 Stability criterion

The following result is of fundamental importance in system theory. It asserts the possibility of establishing stability or asymptotic stability of equilibrium points without explicitly computing trajectories.

Theorem (Lyapunov). Let $x_e = 0$ be an equilibrium point for the system (1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite continuously differentiable function.

1. If $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ is negative semi-definite, then x_e is stable.
2. If $\dot{V}$ is negative definite, then x_e is asymptotically stable.

Proof. Suppose that $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ is negative semi-definite. Given $\varepsilon > 0$, consider the closed ball $\bar{\mathcal{B}}(0, \varepsilon)$. Since its boundary $\mathcal{S}(0, \varepsilon)$ is compact (closed and bounded) and V is continuous, V admits a minimum m on $\mathcal{S}(0, \varepsilon)$ by Weierstrass's theorem. Such minimum is positive because V is positive definite:

$$\min_{\{x : \|x\| = \varepsilon\}} V(x) = m > 0$$

Since V is continuous, in particular at the origin, there exists a $\delta > 0$ such that

$$\bar{x} \in \mathcal{B}(0, \delta) \Rightarrow |V(x) - V(0)| = V(x) < m$$

We claim that this δ is the "right δ" required in the definition of stability, so that any trajectory starting from $\mathcal{B}(0, \delta)$ never exits $\mathcal{B}(0, \varepsilon)$. Choose indeed $\bar{x} \in \mathcal{B}(0, \delta)$ as the initial condition for (1), and for the sake of contradiction suppose that the trajectory $\varphi(t; 0, \bar{x})$ is not entirely contained in the ball $\mathcal{B}(0, \varepsilon)$. Then there exists a time T in which the trajectory intersects the boundary of $\bar{\mathcal{B}}(0, \varepsilon)$, i.e. $V(\varphi(T; 0, \bar{x})) \geq m$. But the derivative of V with respect to time, that is $\dot{V}$, is negative semi-definite, hence V is non-increasing along the corresponding trajectory (that is, $V(\varphi(T; 0, \bar{x})) \leq V(\bar{x})$). Therefore,

$$m \leq V(\varphi(T; 0, \bar{x})) \leq V(\bar{x}) < m$$

which is a contradiction. Hence, the trajectory is contained in $\mathcal{B}(0, \varepsilon)$. Given $\epsilon > 0$, we have constructed a $\delta > 0$ such that if $\bar{x} \in \mathcal{B}(0, \delta)$ then $\varphi(t; 0, \bar{x}) \in \mathcal{B}(0, \varepsilon)$ for all $t \geq 0$. Hence, 0 is a stable equilibrium point.

Suppose now that $\dot{V}$ is negative definite. Of course, this implies that $\dot{V}$ is also negative semi-definite, hence the first property in the definition of asymptotic stability is trivially satisfied. This means that, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\bar{x} \in \mathcal{B}(0, \delta)$ then $\varphi(t; 0, \bar{x}) \in \mathcal{B}(0, \varepsilon)$ for all $t \geq 0$.
We claim that $\lim_{t \to +\infty} \varphi(t; 0, \bar{x}) = 0$ or, more explicitly that for all ε' such that $0 < \varepsilon' < \varepsilon$, there exists a certain time T such that $\varphi(t; 0, \bar{x}) \in \mathcal{B}(0, \varepsilon')$ for all $t \geq T$. Indeed, in view of stability and time invariance, for all $\varepsilon' > 0$ there exists a $\delta' > 0$ such that, if $x(T) \in \mathcal{B}(0, \delta')$, then $\varphi(t; T, x(T)) \in \mathcal{B}(0, \varepsilon')$ for all $t \geq T$. Hence, we just need to prove that there exists T such that $x(T) \in \mathcal{B}(0, \delta')$.
For the sake of contradiction, suppose that this is not the case. Then, for all $t \geq 0$ we have

$$\varphi(t; 0, \bar{x}) \in \bar{\mathcal{B}}(0, \varepsilon) \setminus \mathcal{B}(0, \delta')$$

Since $\bar{\mathcal{B}}(0, \varepsilon) \setminus \mathcal{B}(0, \delta')$ is compact, and $\dot{V}$ is continuous and negative definite, $\dot{V}$ attains a negative maximum $-\mu$ there. Hence,

$$\dot{V}(x) \leq -\mu \text{ if } x \in \bar{\mathcal{B}}(0, \varepsilon) \setminus \mathcal{B}(0, \delta')$$

and finally

$$\begin{aligned} V(\varphi(t; 0, \bar{x})) &= V(\bar{x}) + \int_0^t \dot{V}(\varphi(\tau; 0, \bar{x})) d\tau \\ &= V(\bar{x}) - \mu t \end{aligned}$$

Letting $t \to +\infty$ we obtain a contradiction, because $V(x) \geq 0$ for all $x \in \bar{\mathcal{B}}(0, \varepsilon)$, but the right-hand side tends to $-\infty$. Therefore there must exist T such that $x(T) \in \mathcal{B}(0, \delta')$. This proves the theorem. $\square$

Pay attention to the fact that Lyapunov's theorem *assumes* the existence of a Lyapunov function with negative (semi-)definite Lie derivative, but does not provide any method to construct one. In other words, proving the stability of an equilibrium is still not a straightforward task, unless we find such a function by other means. In certain cases, such as linear systems, Lyapunov functions arise naturally, but in general their construction is an open problem.

3 Asymptotic stability of linear systems

Now we apply Lyapunov's theorem to the analysis of time-invariant linear systems. We start with an algebraic result:

Theorem. Let $A \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

1. all the eigenvalues of A have negative real part;
2. for all matrices $Q = Q^\top > 0$ there exists an unique solution $P = P^\top > 0$ to the following (Lyapunov) equation:
$$A^\top P + PA + Q = 0 \tag{2}$$

We will just prove the "only if" part, namely:

Proof $(1 \Rightarrow 2)$. Suppose that A has eigenvalues with negative real part. Define

$$P := \int_0^{+\infty} e^{A^\top t} Q e^{At} \, dt$$

Since the elements of the integrand matrix are all linear combinations of functions of the form $t^k e^{\alpha t}$ where α has negative real part (the reader can verify this considering the Jordan form of A), the integral exists and is finite. Now we verify that such P is a solution to (2). Indeed,

$$\begin{aligned}
A^\top P + PA &= \int_0^{+\infty} (A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A) \, dt \\
&= \int_0^{+\infty} \frac{d}{dt} \left(e^{A^\top t} Q e^{At} \right) \, dt \\
&= \left[e^{A^\top t} Q e^{At} \right]_0^{+\infty} \\
&= -Q
\end{aligned}$$

(since $\lim_{t \to +\infty} e^{A^\top t} Q e^{At} = 0$). Moreover, the solution is unique; indeed, let P_1 and P_2 be any two solutions:

$$\begin{aligned}
A^\top P_1 + P_1 A + Q = 0 \\
A^\top P_2 + P_2 A + Q = 0
\end{aligned}$$

Subtracting the second equality from the first,

$$A^\top (P_1 - P_2) + (P_1 - P_2) A = 0$$

therefore

$$\begin{aligned}
0 &= e^{A^\top t} \left(A^\top (P_1 - P_2) + (P_1 - P_2) A \right) e^{At} \\
&= e^{A^\top t} A^\top (P_1 - P_2) e^{At} + e^{A^\top t} (P_1 - P_2) A e^{At} \\
&= \frac{d}{dt} \left(e^{A^\top t} (P_1 - P_2) e^{At} \right)
\end{aligned}$$

In other terms, $e^{A^\top t}(P_1 - P_2)e^{At}$ is constant, hence for all t it has the same value that it assumes at $t = 0$:

$$e^{A^\top t}(P_1 - P_2)e^{At} = P_1 - P_2$$

Finally, letting $t \to +\infty$ on the left-hand side, we obtain $P_1 - P_2 = 0$, so that any two solutions coincide; uniqueness follows. $\square$

Consider now the time-invariant linear system

$$\dot{x}(t) = A\, x(t) \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$. We apply Lyapunov's theorem to verify a result that we already know from modal analysis:

Theorem. If all the eigenvalues of A have negative real part, then the system (3) is asymptotically stable.

Proof. Since the eigenvalues of A have negative real part, there exists a positive definite solution P to the Lyapunov equation

$$A^\top P + PA + I = 0$$

Let us choose, as a Lyapunov function, the quadratic form $V(x) = x^\top P x$, which is of course a positive definite function. Its Lie derivative is

$$\begin{aligned}\dot{V}(x) &= \dot{x}^\top P x + x^\top P \dot{x} \\ &= (Ax)^\top P x + x^\top P A x \\ &= x^\top (A^\top P + PA) x \\ &= x^\top (-I) x \\ &= -\|x\|^2\end{aligned}$$

which is negative definite. Hence, the system is asymptotically stable. $\square$

When all the eigenvalues of A have negative real part, we will call A a *stability matrix.*

4 Stability analysis via linearization

Suppose that $x_e = 0$ is an equilibrium point for (1), and that the function f in (1) is continuously differentiable. Then we can linearize the system around 0, namely consider the time-invariant linear system

$$\dot{x}(t) = A\, x(t) \tag{4}$$

where $A = \frac{\partial f}{\partial x}(0) \in \mathbb{R}^{n\times n}$.
It turns out that, if A has eigenvalues with negative real part, something interesting can be said not just on (4) but also on the nonlinear system. We have the following result:

Theorem Let $x_e = 0$ be an equilibrium point for (1), and let $f \in C^1(\mathbb{R}^n)$. If $A = \frac{\partial f}{\partial x}(0)$ is a stability matrix, then x_e is an asymptotically stable equilibrium point for the system (1).

Proof. Suppose that A in (4) is a stability matrix. Consider the Taylor expansion of f around its equilibrium point:

$$\begin{aligned} f(x) &= f(0) + Ax + \sigma(x)\,\|x\| \\ &= Ax + \sigma(x)\,\|x\| \end{aligned}$$

where $\lim_{\|x\|\to 0} \sigma(x) = 0$. Since A is a stability matrix, the equation

$$A^\top P + PA + I = 0$$

admits a solution $P = P^\top > 0$. Consider the (positive definite) quadratic form

$$V(x) := x^\top P x.$$

We will apply again Lyapunov's theorem using V as the Lyapunov function. We have indeed

$$\begin{aligned} \dot{V}(x) &= \nabla V(x) \cdot f(x) \\ &= 2x^\top P(Ax + \sigma(x)\,\|x\|) \\ &= x^\top (A^\top P + PA)x + 2x^\top P\sigma(x)\,\|x\| \\ &= -x^\top x + 2x^\top P\sigma(x)\,\|x\| \\ &= \|x\|^2 \left(-1 + \frac{2x^\top P\sigma(x)}{\|x\|}\right) \end{aligned}$$

But now, from Schwartz's inequality,

$$\begin{aligned} \left|2x^\top P\sigma(x)\right| &= |\langle x, 2P\sigma(x)\rangle| \\ &\le \|x\|\,\|2P\sigma(x)\| \\ &\le 2\,\|x\|\,\|P\|\,\|\sigma(x)\| \end{aligned}$$

Therefore the term $\frac{2x^\top P\sigma(x)}{\|x\|}$ also tends to 0 as $x \to 0$, there exists $\varepsilon > 0$ such that $\dot{V}(x) < 0$ for all $x \in \mathcal{B}(0,\varepsilon) \setminus \{0\}$, and $\dot{V}$ is negative definite. Invoking Lyapunov's theorem, we can conclude that the origin is an asymptotically stable equilibrium point of (1). $\square$

We mention without proof another result:

Theorem Let $x_e = 0$ be an equilibrium point for (1), and let $f \in C^1(\mathbb{R}^n)$. If the matrix $A = \frac{\partial f}{\partial x}(0)$ has at least one eigenvalue with (strictly) positive real part, then the origin is an *unstable* equilibrium point for both the linearized system (4) and the nonlinear system (1).

Note that we cannot say anything when we just know that the eigenvalues of A lie in the *closed* left-hand plane, that is, nothing can be said if there are eigenvalues on the imaginary axis. For instance, a linear system with all its eigenvalues at 0 can be either stable or unstable. (Consider e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ vs. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.)

Even when the linearized system is stable, this does *not* imply stability of the nonlinear system. As a counterexample, consider the equilibrium point $x_e = 0$ for the nonlinear scalar systems $\dot{x}(t) = -x^2(t)$ and $\dot{x}(t) = x^2(t)$. By separation of variables, their general solutions are respectively $x(t) = \frac{1}{1/x_0+t}$ and $x(t) = \frac{1}{1/x_0-t}$. Hence, it is easily seen that the origin is an asymptotically stable equilibrium for the first system, and an unstable equilibrium for the second (because the solution diverges in finite time). Nevertheless, the linearization of both the systems around the origin yields $\dot{x}(t) = 0$, which is just stable.