Attaining Mean Square Boundedness of a Marginally Stable Stochastic Linear System With a Bounded Control Input

Federico Ramponi, Debasish Chatterjee, Andreas Milias-Argeitis, Peter Hokayem, and John Lygeros

***Abstract*—We construct control policies that ensure bounded variance of a noisy marginally stable linear system in closed-loop. It is assumed that the noise sequence is a mutually independent sequence of random vectors, enters the dynamics affinely, and has bounded fourth moment. The magnitude of the control is required to be of the order of the first moment of the noise, and the policies we obtain are simple and computable.**

***Index Terms*—Bounded controls, linear systems, stochastic stability.**

I. Introduction

Stabilization of stochastic linear systems with bounded control inputs has attracted considerable attention over the years. This is due to the fact that incorporating bounds on the control is of paramount importance in practical applications; suboptimal control strategies such as receding-horizon control [1], [2], and rollout algorithms [3], among others, were designed to incorporate such constraints with relative ease, and have become widespread in applications. However, the following question remains open: *when is a linear system with possibly unbounded additive stochastic noise globally stabilizable with bounded controls?* In this article, we provide sufficient conditions that give a positive answer to this question.

Bounded input control has a rich and important history in the control literature [4]–[8]. The deterministic version of the bounded input stabilization problem was solved completely in a series of articles [4], [5] culminating in [6]. It was demonstrated in [6] that global asymptotic stabilization of a discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t \tag{$*$}$$

with bounded feedback controls is possible if and only if the transition matrix has spectral radius at most 1, and the pair (A, B) is stabilizable with arbitrary controls. Extensions to the output feedback case have appeared in [9], [10]. In the presence of affine stochastic noise the linear system $(*)$ becomes

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{$**$}$$

where $(w_t)_{t\in\mathbb{N}_0}$ is a collection of independent (but not necessarily identically distributed) random vectors with possibly inter-dependent components at each time t. In this setting one fundamental question of closed-loop stability is the following: Does there exist a control policy such that, given an arbitrary initial state x_0, the sequence $(\|x_t\|^2)_{t\in\mathbb{N}_0}$ generated by the closed-loop system is bounded in expectation?

Manuscript received October 16, 2009; revised February 08, 2010, June 07, 2010; accepted June 07, 2010. Date of publication June 28, 2010; date of current version October 06, 2010. This work supported in part by the Swiss National Science Foundation under Grant 200021-122072. Recommended by Associate Editor H. Zhang.

The authors are with the Automatic Control Laboratory, ETL I28 ETH Zürich, Zürich 8092, Switzerland (e-mail: ramponif@control.ee.ethz.ch; ,chatterjee@control.ee.ethz.ch; milias@control.ee.ethz.ch; hokayem@control.ee.ethz.ch; lygeros@control.ee.ethz.ch).

Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2010.2054850

Ensuring mean-square boundedness of $(**)$ is clearly not possible in the presence of an arbitrary noise; it is necessary to assume, at least, that the noise has bounded variance.[1] Going beyond this necessary condition and using standard Foster–Lyapunov techniques [11], it is not difficult to establish mean-square boundedness of such a system with bounded controls under the assumption that A is Schur stable, i.e., all eigenvalues of A are contained in the interior of the unit disk. It is also not difficult to see that ensuring a mean-square bound in closed-loop with the aid of bounded controls is not possible if A is unstable. However, to the best of our knowledge, there is no *conclusive* proof that a closed-loop mean-square bound can be ensured with a marginally stable A. Results in this direction were reported in [8] and [12], where the authors proposed a controller comprising a variable gain linear feedback followed by a fixed saturation and an argument that for low enough values of the gain this controller leads to bounded state variance for marginally stable linear systems. Here we propose a different control scheme derived from a fixed gain linear controller followed by saturation. The saturation function utilized in the construction of our controller is effective outside a small region close to the origin, whereas the saturation function in [8] is (by construction) generally effective very far from the origin. This difference can have a major impact on the relative performance of the two policies as illustrated through an example in Section IV.

We develop easily computable bounded control policies for the case when A is marginally stable and (A, B) is stabilizable. Our policy is not in general stationary and is selected from the class of finite k-history-dependent and/or non-stationary policies. In the case when A is orthogonal, it turns out that if the system is reachable in one step (i.e., $\mathrm{rank}B = \mathrm{the\ dimension\ of\ the\ state\ space}$), we do get stationary feedback policies. In the more general case when the system $(*)$ is reachable in k steps (with arbitrary controls), we propose a feedback policy for a sub-sampled system derived from the original one, which, for the actual system, turns out to be a k-history-dependent policy. In fact, in this case we realize our policy as successive concatenations of a fixed k-length policy. In the most general situation we propose a k-history-dependent policy, where k is now the reachability index of the particular subsystem of (A, B) for which the dynamics matrix is orthogonal. In all these cases, the length of the policy is at most equal to the dimension of the state space; memory requirements for even the most general case are, therefore, modest.

Note that we do *not* assume that the noise is white. For our purposes the requirements on the noise are rather general, namely, the fourth moment of the noise should be uniformly bounded, and the noise vectors should be independent of each other (identical distribution at each time is not assumed). In particular, we do *not* assume Gaussian structure of the noise. It turns out that to ensure stabilization we need the controller to be sufficiently strong, in the sense that the control input norm bound should be bigger than a uniform bound on the first moment of the noise.

Section II contains a precise statement of our result under the most general hypotheses (A marginally stable and (A, B) stabilizable), and a brief sketch of the proof. In Section III, after some preliminary material, we prove the attainability of bounded second moment for a random walk, then we generalize the result under weaker and weaker hypotheses, finally culminating in the proof of the main theorem of Section II. The final Section IV presents a numerical example illustrating our results.

[1]For instance, if the noise has a spherically symmetric Cauchy distribution on $\mathbb{R}^d$, then given any initial condition $x_0 \in \mathbb{R}^d$, the second moment of x_1 does not even exist. Similarly, if the second moment of the noise becomes unbounded with time, it is not possible to control the second moment of the process $(x_t)_{t\in\mathbb{N}_0}$.

II. Main Result

A. Statement of the Theorem

Consider the discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 = x, \quad t \in \mathbb{N}_0 \tag{1}$$

where the following hold: $x \in \mathbb{R}^d$ is given; the state x_t at time t takes values in $\mathbb{R}^d$; $A \in \mathbb{R}^{d\times d}$, all the eigenvalues of A lie in the closed unit circle, and those eigenvalues λ such that $|\lambda| = 1$ have equal algebraic and geometric multiplicities; $B \in \mathbb{R}^{d\times m}$, and the control u_t at time t takes values in $\mathbb{R}^m$; $(w_t)_{t\in\mathbb{N}_0}$ is an $\mathbb{R}^d$-valued random process.

Our objective is to synthesize a k-history-dependent control policy[2] $\pi = (\pi_t)_{t\in\mathbb{N}_0}$, consisting of successive concatenations of a k-length sequence of maps $\tilde{\pi}_{0:k-1} := [\tilde{\pi}_0, \cdots, \tilde{\pi}_{k-1}]$, $\tilde{\pi}_i : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ for $i = 0, \ldots, k-1$, such that $\pi_t : \mathbb{R}^{d\times k} \longrightarrow \mathbb{R}^m$ is measurable, $u_t := \pi_t(x_t, x_{t-1}, \ldots, x_{t-k+1})$, the sequence $(u_t)_{t\in\mathbb{N}_0}$ is bounded, and the state of the closed-loop system

$$\begin{aligned} x_{t+1} &= Ax_t + B\pi_t(x_t, \ldots, x_{t-k+1}) + w_t, \\ x_0 &= x, \qquad t \in \mathbb{N}_0 \end{aligned} \tag{2}$$

has bounded second-order moment. (To simplify the notation, we fix $x_{-k+1} = \cdots = x_{-1} = x_0$.) The following is our main result:

Theorem 1: Consider the system (1). Suppose that the pair (A, B) is *stabilizable*, and that $\sup_{t\in\mathbb{N}_0} \mathsf{E}[\|w_t\|^4] < \infty$. Then there exist an $R > 0$ and a deterministic k-history-dependent policy $(\pi_t)_{t\in\mathbb{N}_0}$, with $k \le d$ and $\|\pi_t(\cdot)\| \le R$ for every t, such that

(P1) for every fixed $x \in \mathbb{R}^d$ the solution $(x_t)_{t\in\mathbb{N}_0}$ to (2) satisfies $\sup_{t\in\mathbb{N}_0} \mathsf{E}_x[\|x_t\|^2] < \infty$;

(P2) in the absence of the random noise the origin is asymptotically stable for the closed-loop system.

B. Sketch of the Proof

Our proof is built in a series of steps, moving from simpler to progressively more complex systems. The starting point is the d-dimensional random walk $x_{t+1} = x_t + u_t + w_t$. In this case we employ the main result of [13] to design a policy that guarantees mean-square boundedness of the closed-loop system. We then consider the system $x_{t+1} = Ax_t + Bu_t + w_t$, where u_t is a d-dimensional control input, $\operatorname{rank} B = d$, and A is orthogonal. With the help of a time-varying injective linear transformation this case is reduced to the d-dimensional random walk. The third case is when $u_t \in \mathbb{R}^m$ and A is orthogonal. This is reduced to the second case above with the aid of an injective linear transformation derived from the reachability matrix of the pair (A, B). Finally, the general case when A is just stable and (A, B) stabilizable is reduced to the third case by observing that A acts as an orthogonal map on the invariant subspace corresponding to the eigenvalues that lie on the unit circle.

Arguments for establishing mean-square boundedness of stochastic dynamical systems typically rely on L_1-boundedness of a Lyapunov-like function for the system. The latter can be established in at least three different ways: The first is via the classical Foster-Lyapunov drift-conditions [11], [14] and its various refinements; this is the approach followed in [8]. The second is via excursion-theoretic analysis [15] that relies primarily on the existence of certain supermartingales as long as the process is outside some bounded set. The third is via martingale inequalities [13], and this is the approach pursued in this article.

[2]See Section III-A for definitions of policies.

III. Proof of the Main Result

A. Preliminaries

If $(y_t)_{t\in\mathbb{N}_0}$ is a random process on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, taking values in some Euclidean space, we let $\mathsf{E}_x[\varphi(y_s; s = 0, 1, \ldots, t)]$ denote the conditional expectation of a measurable mapping φ of the process up to time t, given the initial condition $y_0 = x$; in particular we define the n-th moment of y_t as $\mathsf{E}_x[\|y_t\|^n]$.[3] We denote conditional expectation given a sub-σ-algebra $\mathfrak{F}'$ of $\mathfrak{F}$ as $\mathsf{E}[\cdot|\mathfrak{F}']$. For $r > 0$ let $\operatorname{sat}_r : \mathbb{R}^d \longrightarrow \mathcal{B}_r$ be defined by $\operatorname{sat}_r(y) := y$ if $y \in \mathcal{B}_r$ and $\operatorname{sat}_r(y) := ry/\|y\|$ otherwise.[4] Given matrices $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ we define the k-step reachability matrix $\mathcal{R}_k := [B \; AB \; \cdots \; A^{k-1}B]$.

We specialize the general definition of a policy [16, Ch. 2] to our setting. A policy $\pi := (\pi_t)_{t\in\mathbb{N}_0}$ is a sequence of measurable maps $\pi_t : \mathbb{R}^{d\times k} \longrightarrow \mathbb{R}^m$ for some $k \in \mathbb{N}$, such that the control at time t is $\pi_t(x_t, x_{t-1}, \ldots, x_{t-k+1})$. The policy $\pi = (\pi_t)_{t\in\mathbb{N}_0}$ we have defined is also known as a *deterministic k-history-dependent policy* in the literature. A special case of these policies is a *deterministic feedback policy* or simply a *feedback* if $k = 1$ in the definition of a deterministic history-dependent policy. Under deterministic feedback policies the closed-loop system is Markovian [16, Proposition 2.3.5]. A further special case is when $\pi_t = f$, a fixed measurable mapping $f : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ for $t \in \mathbb{N}_0$; this is known as a *stationary feedback policy*.

Lemma 2: Let $B_1, \cdots, B_k$ be $d \times m$ matrices, $M := [B_1 \; \cdots \; B_k]$, and σ_d denote the minimum singular value of M. If $\operatorname{rank} M = d$, then for all $r > 0$ every vector $v \in \mathbb{R}^d$ belonging to $\mathcal{B}_r$ can be expressed as $v = \sum_{i=1}^k B_i u_i$, with $u_i \in \mathbb{R}^m$ and $\|u_i\| \le r\sigma_d^{-1}$. In particular, if $B \in \mathbb{R}^{d\times d}$ and $\operatorname{rank} B = d$, then every vector $v \in \mathbb{R}^d$ belonging to $\mathcal{B}_r$ can be expressed as $v = Bu$, where $u \in \mathbb{R}^d$, $\|u\| \le r\sigma_d^{-1}$.

Proof: $\operatorname{rank} M = d$ implies that $km \ge d$. Hence, $M = [B_1 \; \cdots \; B_k] \in \mathbb{R}^{d\times km}$ is a "flat" matrix. Let $M = USV^{\mathrm{T}} = U[\Sigma \;\; 0]V^{\mathrm{T}}$ be a singular value decomposition of M, where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_d)$. Since M has full rank, the matrix Σ is invertible. Hence every vector $v \in \mathbb{R}^d$ can be expressed as $v = Mu$, where $u = M^+v$ and $M^+ = V\left[\begin{smallmatrix}\Sigma^{-1}\\ 0\end{smallmatrix}\right]U^{\mathrm{T}} \in \mathbb{R}^{km\times d}$ is the Moore-Penrose pseudoinverse of M. Since U, V are orthogonal, for any $\rho > 0$ we have $\inf_{\|u\|=\rho} \|Mu\| = \inf_{\|V^{\mathrm{T}}u\|=\rho} \|U[\Sigma \;\; 0]V^{\mathrm{T}}u\| = \inf_{\|v\|=\rho} \|\Sigma v\| = \rho\sigma_d$. Hence, the image of $\mathcal{B}_\rho$ under M contains $\mathcal{B}_{\rho\sigma_d}$, and if we choose $\rho = r\sigma_d^{-1}$, then the image of $\mathcal{B}_\rho$ under M contains $\mathcal{B}_r$. Notice that σ_d^{-1} is also the greatest singular value of M^+, and indeed we have $\sup_{\|v\|=r} \|M^+v\| = \sup_{\|U^{\mathrm{T}}v\|=r} \|V\left[\begin{smallmatrix}\Sigma^{-1}\\ 0\end{smallmatrix}\right]U^{\mathrm{T}}v\| = \sup_{\|\nu\|=r} \|\Sigma^{-1}\nu\| = r\sigma_d^{-1}$. Summing up, every $v \in \mathcal{B}_r$ can be expressed as $v = Mu$, where $u \in \mathbb{R}^{km}$ and $\|u\| \le r\sigma_d^{-1}$. It remains to notice that u can be partitioned according to the partition of M, that is $v = Mu = [B_1 \; B_2 \; \cdots \; B_k][u_1^{\mathrm{T}} \; \cdots \; u_k^{\mathrm{T}}]^{\mathrm{T}} = \sum_{i=1}^k B_i u_i$ and the bound $\|u\| \le r\sigma_d^{-1}$ implies $\|u_i\| \le r\sigma_d^{-1}$ for all $i = 1 \cdots k$. ■

B. The d-Dimensional Random Walk

At the core of our proof is the d-dimensional random walk

$$x_{t+1} = x_t + u_t + w_t, \qquad x_0 = x, \quad t \in \mathbb{N}_0 \tag{3}$$

with the state $x_t \in \mathbb{R}^d$, the control $u_t \in \mathbb{R}^d$ with $\|u_t\| \le r$ for some $r > 0$, the noise process $(w_t)_{t\in\mathbb{N}_0}$ satisfies the following assumption:

[3]Let $\mathbb{N}_0$ be the set of nonnegative integers $\{0, 1, 2, \ldots\}$. The standard 2-norm on Euclidean spaces is denoted by $\|\cdot\|$ and the absolute value on $\mathbb{R}$ by $|\cdot|$. In a Euclidean space we denote by $\mathcal{B}_r$ the closed Euclidean ball of radius r centered at the origin.

[4]Note that $\operatorname{sat}_r(\cdot)$ is *not* the component-wise saturation function.

Assumption 3:

- $(w_t)_{t\in\mathbb{N}_0}$ are mutually independent d-dimensional random vectors (not necessarily identically distributed);
- $\mathsf{E}[w_t w_t^{\mathrm{T}}] = Q_t$ for all $t \in \mathbb{N}_0$;
- there exists $C_4 > 0$ such that $\mathsf{E}[\|w_t\|^4] \le C_4$ for all $t \in \mathbb{N}_0$. $\diamondsuit$

Let $C_1 := \sup_{t\in\mathbb{N}_0} \mathsf{E}[\|w_t\|]$; this is well-defined because by Jensen's inequality we have $C_1 \le \sqrt[4]{C_4}$. Let $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$ be the natural filtration of the system (3). Our proof of Theorem 1 relies on the following (immediate) adaptation of the fundamental result [13, Theorem 1].

Proposition 4: Let $(\xi_t)_{t\in\mathbb{N}_0}$ be a sequence of nonnegative random variables on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, and let $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$ be any filtration to which $(\xi_t)_{t\in\mathbb{N}_0}$ is adapted. Suppose that there exist constants $b > 0$, and $J, M < \infty$, such that $\xi_0 \le J$, and for all t

$$\mathsf{E}[\xi_{t+1} - \xi_t | \mathfrak{F}_t] \le -b \text{ on the event } \{\xi_t > J\}, \text{ and} \tag{4}$$

$$\mathsf{E}[|\xi_{t+1} - \xi_t|^4 | \xi_0, \ldots, \xi_t] \le M. \tag{5}$$

Then there exists a constant $c = c(b, J, M) > 0$ such that $\sup_{t\in\mathbb{N}_0} \mathsf{E}[\xi_t^2] \le c$.

Lemma 5: Consider the system (3), and define $\xi_t := \|x_t\|$, $t \in \mathbb{N}_0$. There exists a constant $b > 0$ such that for any $r > C_1$ condition (4) holds in closed-loop with the control $u_t = -\mathrm{sat}_r(x_t)$.

Proof: Fix $t \in \mathbb{N}_0$ and $r > C_1$. We have $\mathsf{E}[\xi_{t+1} - \xi_t|\mathfrak{F}_t] = \mathsf{E}[\|x_{t+1}\| - \|x_t\||\mathfrak{F}_t] = \mathsf{E}[\|x_t + u_t + w_t\| - \|x_t\||\mathfrak{F}_t] = \mathsf{E}[\|x_t - \mathrm{sat}_r(x_t) + w_t\| - \|x_t\||\mathfrak{F}_t] \le \mathsf{E}[\|x_t - \mathrm{sat}_r(x_t)\| + \|w_t\| - \|x_t\||\mathfrak{F}_t]$. Let $J = r$ and $b := r - C_1$. On the set $\{\|x_t\| > J\}$ we have $\|x_t - \mathrm{sat}_r(x_t)\| - \|x_t\| = -r$. From the above we get, on the set $\{\|x_t\| > J\}$, $\mathsf{E}[\xi_{t+1} - \xi_t|\mathfrak{F}_t] \le -r + \mathsf{E}[\|w_t\|] \le -b$, where b is positive by our hypothesis. The assertion follows. ∎

Lemma 6: Consider the system (3) and define $\xi_t := \|x_t\|$, $t \in \mathbb{N}_0$. Then for the closed-loop system with $u_t = -\mathrm{sat}_r(x_t)$ there exists a constant $M = M(C_4) > 0$ such that (5) holds.

Proof: Fix $r > C_1$. Applying the triangle inequality successively, we have $|\xi_{t+1} - \xi_t|^4 = |\|x_{t+1}\| - \|x_t\||^4 \le \|x_{t+1} - x_t\|^4 = \|u_t + w_t\|^4 \le (r + \|w_t\|)^4$, and $\mathsf{E}[|\xi_{t+1} - \xi_t|^4|\xi_0, \ldots, \xi_t] \le \mathsf{E}[(r + \|w_t\|)^4|\xi_0, \ldots, \xi_t] = \mathsf{E}[(r+\|w_t\|)^4]$. Since the fourth moment of w_t is uniformly bounded, expanding the right-hand side above and applying Jensen's inequality shows that there exists some $M = M(r, C_4) > 0$ such that $\mathsf{E}[(r + \|w_t\|)^4] \le M$. The assertion follows. ∎

Proposition 7: For $r > 0$ consider the system (3) under the deterministic stationary feedback policy $u_t = -\mathrm{sat}_r(x_t)$

$$x_{t+1} = x_t - \mathrm{sat}_r(x_t) + w_t, \qquad x_0 = x, \quad t \in \mathbb{N}_0. \tag{6}$$

Then for every $r > C_1$ the system (6) satisfies $\sup_{t\in\mathbb{N}_0} \mathsf{E}_x[\|x_t\|^2] \le c$ for some $c = c(x, r, C_1) < \infty$.

Proof: Let $r = C_1 + b$ for some $b > 0$ and $J := \max\{r, \|x\|\}$. Lemma 5 guarantees that (4) holds, and Lemma 6 shows that there exists an $M > 0$ such that (5) holds. The assertion now is an immediate consequence of Proposition 4. ∎

C. The Case of A Orthogonal

Next we establish part (P1) of the main theorem in the particular case of A being *orthogonal*.

Lemma 8: Consider the system $y_{t+1} = Ay_t + u_t + w_t, y_0 = x$, where y_t and u_t take values in $\mathbb{R}^d$, A is orthogonal, and $(w_t)_{t\in\mathbb{N}_0}$ satisfies Assumption 3. There exist a constant $r > 0$ and a deterministic stationary policy $\pi = (f, f, \cdots)$ such that $\|f(y)\| \le r$ for all $y \in \mathbb{R}^d$ and $t \in \mathbb{N}_0$, and the closed-loop system

$$y_{t+1} = Ay_t + f(y_t) + w_t \tag{7}$$

under this policy satisfies $\sup_{t\in\mathbb{N}_0} \mathsf{E}_x[\|y_t\|^2] < \infty$.

Proof: Consider the process $(z_t)_{t\in\mathbb{N}_0}$ defined by $z_t := (A^{\mathrm{T}})^t y_t$. The second moment of z_t is the same as that of y_t due to orthogonality of A: $\mathsf{E}_x[\|z_t\|^2] = \mathsf{E}_x[\|(A^{\mathrm{T}})^t y_t\|^2] = \mathsf{E}_x[y_t^{\mathrm{T}} A^t (A^{\mathrm{T}})^t y_t] = \mathsf{E}_x[y_t^{\mathrm{T}} y_t] = \mathsf{E}_x[\|y_t\|^2]$. Now we have

$$\begin{aligned} z_{t+1} &= (A^{\mathrm{T}})^{t+1} y_{t+1} \\ &= (A^{\mathrm{T}})^t y_t + (A^{\mathrm{T}})^{t+1} u_t + (A^{\mathrm{T}})^{t+1} w_t \\ &= z_t + \bar{u}_t + \bar{w}_t \end{aligned} \tag{8}$$

where the mapping $u_t \mapsto \bar{u}_t := (A^{\mathrm{T}})^{t+1} u_t$ is isometric and invertible, and $(\bar{w}_t)_{t\in\mathbb{N}_0}$ defined by $\bar{w}_t := (A^{\mathrm{T}})^{t+1} w_t$, is a sequence of independent (although in general *not* identically distributed) random vectors, with fourth moment given by $\mathsf{E}[\|\bar{w}_t\|^4] = \mathsf{E}[\|(A^{\mathrm{T}})^{t+1} w_t\|^4] = \mathsf{E}[\|w_t\|^4] \le C_4$. Due to Proposition 7, there exists a constant r such that the closed-loop system (8) under the policy $\bar{u}_t = -\mathrm{sat}_r(z_t) =: \bar{f}(z_t)$ has bounded second moment. Consequently, the original system (7) has bounded second moment under the policy $u_t = A^{t+1}\bar{u}_t = A^{t+1}\bar{f}(z_t) = -A^{t+1}\mathrm{sat}_r((A^{\mathrm{T}})^t y_t) =: f_t(y_t)$. Noting that for any orthogonal matrix A we have $\mathrm{sat}_r(Ay) = A\mathrm{sat}_r(y)$, we arrive at $u_t = f_t(y_t) = -A\mathrm{sat}_r(y_t) =: f(y_t)$, which is indeed a stationary feedback. Moreover, since $\|A\mathrm{sat}_r(y_t)\| \le r$, we have $\|f(y_t)\| \le r$. ∎

In the following we will consider a non-stationary policy obtained by successive concatenations of a k-length policy $(f_0, f_1, \cdots f_{k-1})$ acting on the "sub-sampled" process $(x_{nk})_{n\in\mathbb{N}_0}$. More precisely, our policy has the form $u_t = Bf_{t \bmod k}(x_{(t\div k)k})$ where the "$\div$" symbol denotes integer division and "$\mathbf{mod}$" its remainder. In words, we break the time line into segments of length k, and within each segment we let the controls be given by $f_0, f_1, \cdots f_{k-1}$, applied in this order always to the first state observed in the segment. For example, $x_1 = x_0 + Bf_0(x_0) + w_0, x_2 = x_1 + Bf_1(x_0) + w_1, \ldots, x_k = x_{k-1} + Bf_{k-1}(x_0) + w_{k-1}$, $x_{k+1} = x_k + Bf_0(x_k) + w_k, x_{k+2} = x_{k+1} + Bf_1(x_k) + w_{k+1}$, and so on.

Lemma 9: Consider the system

$$x_{t+1} = Ax_t + Bu_t + w_t, y_0 = x \tag{9}$$

where x_t takes values in $\mathbb{R}^d$, u_t takes values in $\mathbb{R}^m$, A is orthogonal, the pair (A, B) is reachable in k steps (i.e., $\mathrm{rank}\mathcal{R}_k = d$, where $\mathcal{R}_k = [B \; AB \; \cdots \; A^{k-1}B]$), and $(w_t)_{t\in\mathbb{N}_0}$ satisfies Assumption 3. Then there exist a constant $\rho > 0$ and a policy $\pi = (f_0, f_1, \cdots f_{k-1}, f_0, f_1, \cdots)$ such that $\|f_i(x)\| \le \rho$ for all $x \in \mathbb{R}^d$, and the closed-loop system

$$x_{t+1} = Ax_t + Bf_{t \bmod k}\left(x_{(t\div k)k}\right) + w_t \tag{10}$$

under this policy satisfies $\sup_{t\in\mathbb{N}_0} \mathsf{E}_x[\|x_t\|^2] < \infty$.

Proof: Let $\tau \in \mathbb{N}_0$ and consider the evolution of (9) from time τk to time $(\tau + 1)k$

$$\begin{aligned} x_{(\tau+1)k} &= A^k x_{\tau k} + \mathcal{R}_k \left[u_{(\tau+1)k-1}^{\mathrm{T}} \quad \cdots \quad u_{\tau k}^{\mathrm{T}}\right]^{\mathrm{T}} \\ &\quad + \sum_{i=0}^{k-1} A^{k-1-i} w_{\tau k+i} \\ &= \bar{A}x_{\tau k} + \bar{u}_\tau + \tilde{w}_\tau \end{aligned} \tag{11}$$

where $\tilde{w}_\tau := \sum_{i=0}^{k-1} A^{k-1-i} w_{\tau k+i}$ is a random vector with bounded fourth moment. Since $\mathcal{R}_k$ has full rank, we can exploit Lemma 2 setting $M = [B_1 \; \cdots \; B_k] = [B \; \cdots \; A^{k-1}B] = \mathcal{R}_k$, and obtain that for arbitrary $r > 0$, any $\bar{u}_\tau$ in $\mathcal{B}_r$ can be expressed as $\bar{u}_\tau = \sum_{i=0}^{k-1} A^{k-1-i} Bu_{\tau k+i}$, where $\|u_{\tau k+i}\| \le r\sigma_d^{-1}$ and σ_d is the smallest singular value of $\mathcal{R}_k$. In other words, for arbitrary $r > 0$, at time τ we can choose *any* "k-steps" input $\bar{u}_\tau$ such that $\|\bar{u}_\tau\| \le r$; and in doing that we can enforce that each of its components $u_{\tau k} \cdots u_{(\tau+1)k-1}$, which are the actual inputs of system (9) that will be applied in the subsequent time steps $\tau k \cdots (\tau+1)k-1$, will be bounded by $\|u_{\tau k+i}\| \leq r\sigma_d^{-1}$. Now, we know from Lemma 8 that there exists a particular $r > 0$ such that, under the stationary policy $\bar{u}_\tau = f(x_{\tau k}) = -\bar{A}\mathrm{sat}_r(x_{\tau k})$, there exists a constant $c = c(x, C_1, C_4) > 0$ such that $\sup_{\tau\in\mathbb{N}_0} \mathsf{E}_x[\|x_{\tau k}\|^2] \leq c$, that is, the "sub-sampled" system (11) has bounded second moment with "k-steps" input bounded by $\|\bar{u}_\tau\| \leq r$. Therefore, if we choose, as the constant whose existence is claimed in the assertion, $\rho = r\sigma_d^{-1}$, we can attain the same result having the actual input bounded by $\|u_t\| \leq \rho$.

It remains to show that the second-order moment of the state in the intermediate steps (between τk and $(\tau+1)k-1$, say) are also bounded. It follows from the system dynamics and the triangle inequality that for $n = 0, \ldots, k-1$, $\mathsf{E}_x[\|x_{\tau k+n}\|^2] \leq 2(c + n^2r^2\sigma_1(B)^2) + k\max_{n=0,\ldots,k-1}\mathrm{tr}Q_{\tau k+n} \leq 2(c + n^2r^2\sigma_1(B)^2) + k\sqrt{C_4}$, where the last step follows from Jensen's inequality. Since the right-hand side above constitutes a uniform bound, this proves the assertion. ■

Remark 10: The policy for (9) is $[u^{\mathrm{T}}_{(\tau+1)k-1} \cdots u^{\mathrm{T}}_{\tau k}]^{\mathrm{T}} = -\mathcal{R}_k^+\bar{A}\mathrm{sat}_r(x_{\tau k})$. The proof above shows that all the inputs $u_{(\tau+1)k-1}, \cdots, u_{\tau k}$ can be computed at time τk in order to counteract the future effect of the current state, i.e. $\bar{A}x_{\tau k}$, and ignoring the effect of the noise for the following k steps. In the particular case when $B \in \mathbb{R}^{d\times d}$ has full rank, $m = d$, and obviously $k = 1$, the above policy is *stationary*, and in particular it has the form: $u_t = f(x_t) = -B^{-1}A\mathrm{sat}_r(x_t)$. Once again we have $\|u_t\| \leq r\sigma_d^{-1}$, where this time σ_d is the smallest singular value of B. ◃

D. Proof of Theorem 1

Proof: Consider the system (1), with (A, B) stabilizable and $(w_t)_{t\in\mathbb{N}_0}$ with bounded fourth moment. If A is Schur stable (that is, all the eigenvalues of A belong to the interior of the unit disk), the system with zero input has bounded second moment and is asymptotically stable, and there is nothing to prove. Otherwise, there exists a change of base in the state-space that brings the original pair (A, B) to a new pair $(\tilde{A}, \tilde{B})$, where $\tilde{A}$ is in real Jordan form [17, p. 150]. In particular, choosing a suitable ordering of the Jordan blocks, we can ensure that the pair $(\tilde{A}, \tilde{B})$ has the form $\left(\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}, \begin{bmatrix} B_1 \\ B_2\end{bmatrix}\right)$, where A_{11} is Schur stable, and A_{22} has its eigenvalues on the unit circle. Due to the stability hypothesis (the algebraic and geometric multiplicities of the eigenvalues of A_{22} are equal), A_{22} is therefore block-diagonal with elements on the diagonal being either ± 1 or 2×2 rotation matrices. As a consequence, A_{22} is orthogonal. Moreover, since (A, B) is stabilizable, the pair (A_{22}, B_2) must be reachable in a number of steps $k \leq d$ which depends on the dimension of A_{22} and the structure of (A_{22}, B_2), since it contains precisely the modes of A which are not asymptotically stable. Summing up, we can reduce the original system $x_{t+1} = Ax_t + Bu_t + w_t$ to the form $\left[\begin{smallmatrix} x^{(1)}_{t+1} \\ x^{(2)}_{t+1}\end{smallmatrix}\right] = \left[\begin{smallmatrix} A_{11}x^{(1)}_t \\ A_{22}x^{(2)}_t\end{smallmatrix}\right] + \left[\begin{smallmatrix} B_1 \\ B_2\end{smallmatrix}\right]u_t + \left[\begin{smallmatrix} w^{(1)}_t \\ w^{(2)}_t\end{smallmatrix}\right]$, where A_{11} is Schur stable, A_{22} is orthogonal, (A_{22}, B_2) is reachable, and $(\left[\begin{smallmatrix} w^{(1)}_t \\ w^{(2)}_t\end{smallmatrix}\right])_{t\in\mathbb{N}_0}$ is derived from $(w_t)_{t\in\mathbb{N}_0}$ by means of linear transformations. We know that since A_{11} is Schur stable, the noise $(w^{(1)}_t)_{t\in\mathbb{N}_0}$ has bounded second moment, and the control inputs $(u_t)_{t\in\mathbb{N}_0}$ are bounded, then the $x^{(1)}$ sub-system is mean-square bounded under any Markovian control [1, §4]. Therefore, if under some bounded policy the $x^{(2)}$ sub-system is mean-square bounded, the original system will also be mean-square bounded under the same policy. Thus, at least for the proof of (P1), it suffices to restrict our attention to the subsystem described by the pair (A_{22}, B_2). Suppose that this subsystem is reachable in a certain number $k \leq d$ of steps.

The proof of (P1) coincides with the proof of Lemma 9, where we obtain $\rho = r\sigma_d^{-1}$ for $r > C_1$ and σ_d is the smallest singular value of $\mathcal{R}_k$. (Here, $\mathcal{R}_k = [B_2\ A_{22}B_2 \cdots A_{22}^{k-1}B_2]$.) As the control authority required in the claim of the theorem, we choose precisely $R = \rho$.

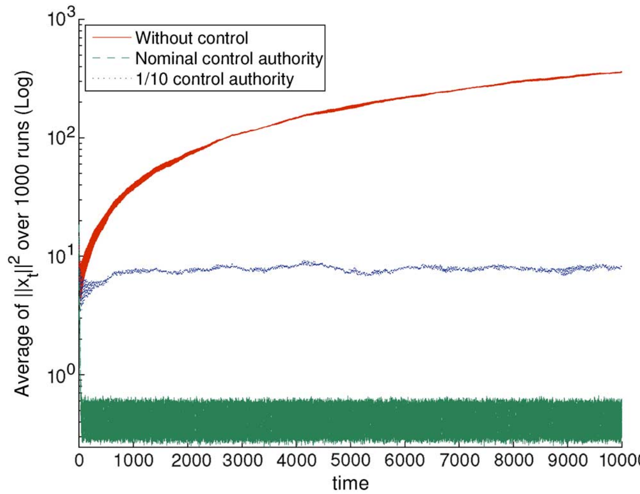

Fig. 1. The policy proposed in this article (with nominal and a tenth of nominal control authority) compared against no control. Empirical average of $\|x_t\|^2$ over 1000 runs.

To prove (P2), notice that for the closed-loop "sub-sampled" system without noise under the policy $u_t = -\mathcal{R}_k^+\bar{A}\mathrm{sat}_r(x^{(2)}_t)$, where $\bar{A} = A_{22}^k$, it holds

$$x^{(2)}_{(\tau+1)k} = \bar{A}x^{(2)}_{\tau k} - \bar{A}\mathrm{sat}_r\left(x^{(2)}_{\tau k}\right). \tag{12}$$

As long as $x^{(2)}_{\tau k}$ is outside $\mathcal{B}_r$, $\|x^{(2)}_{(\tau+1)k}\| = \|x^{(2)}_{\tau k}\| - r$. Hence, in a finite number of steps it must hold $\|x^{(2)}_{\bar{\tau}k}\| < r$. When for some $\bar{\tau}$ we have $\|x^{(2)}_{(\bar{\tau}-1)k}\| < r$, by the definition of $\mathrm{sat}_r(\cdot)$ we have $x^{(2)}_{\bar{\tau}k} = 0$, and consequently $x^{(2)}_{\tau k} = 0$ for all $\tau \geq \bar{\tau}$. Hence, the state of the closed-loop "sub-sampled" system converges to zero in *finite time* for any initial condition. Then, according to the chosen policy, for all $\tau \geq \bar{\tau}$ we have $[u^{\mathrm{T}}_{(\tau+1)k-1} \cdots u^{\mathrm{T}}_{\tau k}]^{\mathrm{T}} = -\mathcal{R}_k^+\bar{A}x^{(2)}_{\tau k} = 0$ and $\bar{u}_\tau = \mathcal{R}_k[u^{\mathrm{T}}_{(\tau+1)k-1} \cdots u^{\mathrm{T}}_{\tau k}]^{\mathrm{T}} = 0$, and consequently, for $\tau \geq \bar{\tau}$ and $\tau k \leq t < (\tau+1)k$ we also have $x^{(2)}_t = 0$, that is, $x^{(2)}_t = 0 \forall t \geq \bar{\tau}k$, which proves (P2) for the subsystem (A_{22}, B_2) of our system (1).

Finally, to extend the result (P2) to the general case (where $A = \mathrm{diag}(A_{11}, A_{22})$), it suffices to note that, since for $t \geq \bar{\tau}k$ it also holds $u_t = 0$, from the time $\bar{\tau}k$ onwards the subsystem (A_{11}, B_1) is in open loop. Since we imposed A_{11} to be Schur stable, the state $x^{(1)}_t$ of the latter converges to zero as $t \to \infty$. This proves the theorem. ■

IV. A Numerical Example

To demonstrate that our nonlinear policy is readily computable and effective, we applied it to the system $x_{t+1} = Ax_t + Bu_t + w_t$, where $A = \begin{bmatrix} \cos\varphi & -\sin\varphi & \cos\varphi - 1 & -0.1 \\ \sin\varphi & \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 0.9\end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0\end{bmatrix}$, with $\varphi = \pi/4$, $x_0 = [2\ 0.5\ -2.5\ 1]^\top$, and (w_t) is Gaussian white noise with mild variance $\Sigma_w = 8.5(10^{-3})\begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 1 & 0 & -1 & 1\end{bmatrix}$. This system is marginally stable and the 3-D subsystem with eigenvalues on the unit circle is reachable in 3 steps, whereas the 1-D Schur-stable subsystem is not reachable at all. The control authority R was chosen equal to 0.99196 according to an estimate of $C_1 = \mathsf{E}[\|w_t\|]$. Fig. 2(a) shows the empirical average of $\|x_t\|^2$ over the 1000 runs, respectively with

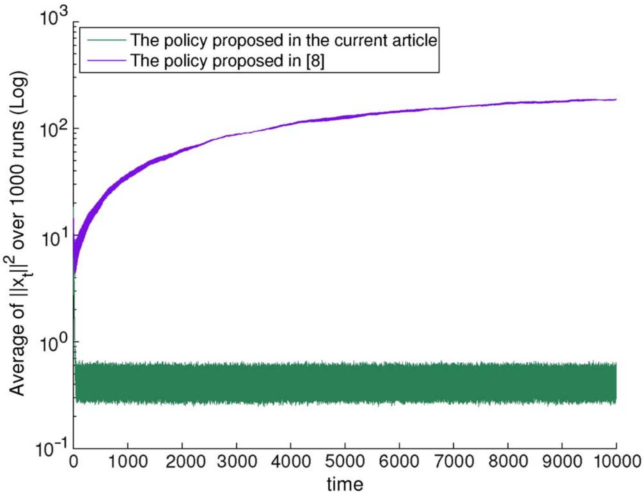

Fig. 2. The policy proposed in this article compared against the one proposed in [8]. Empirical average of $\|x_t\|^2$ over 1000 runs.

disabled control, with the chosen control authority, and with one tenth of the chosen control authority. Fig. 2(b) compares the performance of our policy against the feedback policy proposed in [8], for the same stable linear system. In the case of [8] the saturation is component-wise with $R = 1$, but the linear gain of the controller is tuned in order to match the required design conditions in [8]. Note that the structure of the policy in [8] and the requirement for a low linear gain imply that the saturation is practically inactive within a large region around the origin. As a consequence, the response diverges away from the origin (on an average) until the saturation kicks in at large state values. This is clearly seen in Fig. 2(b) where our policy outperforms that of [8] by two orders of magnitude in terms of the steady state variance.

Note that using our policy smaller values of R are also sufficient to stabilize the system, which leads us to conjecture that if the noise has bounded variance, then *given any arbitrary positive uniform upper-bound* on the norm of the control, there exists a *stationary feedback policy* such that the closed-loop system is mean-square bounded. It appears to us that a proof of this conjecture will require substantially new and nontrivial techniques.

REFERENCES

[1] D. Chatterjee, P. Hokayem, and J. Lygeros, Stochastic receding horizon control with bounded control inputs—A vector-space approach ETH, Zurich, Switzerland, Tech. Rep., 2009.

[2] P. Hokayem, D. Chatterjee, and J. Lygeros, "On stochastic receding horizon control with bounded control inputs," in *Proc. Joint 48th IEEE Conf. Decision Control 28th Chinese Control Conf.*, Shanghai, China, Dec. 2009, pp. 6359–6364.

[3] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, 2nd ed. Belmont, MA: Athena Scientific, 2000, vol. 1.

[4] Y. Yang, H. J. Sussmann, and E. D. Sontag, "Stabilization of linear systems with bounded controls," in *Proc. Nonlin. Control Syst. Design Symp.*, M. Fliess, Ed., 1992, pp. 15–20.

[5] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Autom. Control*, vol. 39, no. 12, pp. 2411–2425, Dec. 1994.

[6] Y. Yang, E. D. Sontag, and H. J. Sussmann, "Global stabilization of linear discrete-time systems with bounded feedback," *Syst. Control Lett.*, vol. 30, no. 5, pp. 273–281, 1997.

[7] Z. Lin, A. Saberi, and A. A. Stoorvogel, "Semi-global stabilization of linear discrete-time systems subject to input saturation via linear feedback—An ARE-based approach," *IEEE Trans. Autom. Control*, vol. 41, no. 8, pp. 1203–1207, Aug. 1996.

[8] A. A. Stoorvogel, A. Saberi, and S. Weiland, "On external semi-global stochastic stabilization of linear systems with input saturation," in *Proc. Amer. Control Conf.*, 2007, pp. 5845–5850.

[9] X. Bao, Z. Lin, and E. D. Sontag, "Finite gain stabilization of discrete-time linear systems subject to actuator saturation," *Automatica*, vol. 36, no. 2, pp. 269–277, Feb. 2000.

[10] Y. Chitour and Z. Linc, "Finite gain l_p stabilization of discrete-time systems subject to actuator saturation: The case of $p = 1$," *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2196–2198, Dec. 2003.

[11] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*. London, U.K.: Springer-Verlag, 1993.

[12] A. A. Stoorvogel, S. Weiland, and A. Saberi, "On stabilization of linear systems with stochastic disturbances and input saturation," in *Proc. 43rd IEEE Conf. Decision Control*, 2004, pp. 3007–3012.

[13] R. Pemantle and J. S. Rosenthal, "Moment conditions for a sequence with negative drift to be uniformly bounded in L^r," *Stochastic Processes Appl.*, vol. 82, no. 1, pp. 143–155, 1999.

[14] S. Foss and T. Konstantopoulos, "An overview of some stochastic stability methods," *J. Oper. Res. Soc. Jpn.*, vol. 47, no. 4, pp. 275–303, 2004.

[15] D. Chatterjee and S. Pal, An Excursion-Theoretic Approach to Stability of Stochastic Hybrid Systems [Online]. Available: http://arxiv.org/abs/0901.2269 2009

[16] O. Hernández-Lerma and J. B. Lasserre, *Discrete-Time Markov Control Processes: Basic Optimality Criteria*, ser. Applications of Mathematics. New York: Springer-Verlag, 1996, vol. 30.

[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

Robust Adaptive Output-Feedback Tracking for a Class of Nonlinear Time-Delayed Plants

Boris Mirkin and Per-Olof Gutman

***Abstract*—Within the model reference adaptive control (MRAC) framework, a continuous adaptive output-feedback control scheme is developed for a class of nonlinear SISO dynamic systems with time delays which is robust with respect to unknown time-varying plant delays and to an external disturbance with unknown bounds. A special form of the Lyapunov-Krasovskii functional with a "virtual" adaptation gain vector is introduced to prove stability.**

***Index Terms*—Nonlinear time-delay systems, robust adaptive control.**

I. INTRODUCTION

The adaptive control technique applied to uncertain systems with time-delays is a research area that is receiving considerable attention during the last few years, see e.g. the recent papers [1]–[17] and the references therein. Many important results have been obtained for linear [1], [3], [4], [8], [10], [12]–[14], [17], switched [16] and nonlinear [5], [6], [9], [13], [14] state or/and input delay plants; based on state [1], [5], [10], [11], [13]–[15], [17] or output [2]–[4], [6]–[8], [12] feedback; with continuous [1], [3]–[9], [17] or discontinuous [2], [10]–[14], [16] control actions.

Manuscript received April 21, 2009; revised September 22, 2009, February 21, 2010, and June 17, 2010; accepted June 20, 2010. Date of publication June 28, 2010; date of current version October 06, 2010. This work was supported by the Israel Science Foundation under Grant 588/07. Recommended by Associate Editor G. Feng.

The authors are with Faculty of Civil and Environmental Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel (e-mail: bmirkin@technion.ac.il; peo@technion.ac.il).

Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2010.2054970