Stable Networked Control Systems With Bounded Control Authority

Peter Hokayem, Debasish Chatterjee, Federico A. Ramponi, and John Lygeros

***Abstract*—We study the stability of a class of networked control systems with hard bounds on the control authority. The plant dynamics are discrete-time, linear, and time-invariant, with stochastic process noise and measurement noise. The controller is designed as a norm-bounded causal history-dependent function of the past outputs perturbed by bounded noise. The resulting control signals are assumed to be transmitted through a lossy channel with packet dropouts. We show that under mild assumptions on the system matrices, the statistics of the process and measurement noise sequences, and the probability of dropouts, it is possible to ensure bounded variance of the system in closed-loop.**

***Index Terms*—Linear Quadratic Gaussian (LQG), networked control systems (NCS).**

I. Introduction

Communication channels have become ubiquitous in control applications such as remotely operated robotic systems [12]. In such applications, measurement and control signals are exchanged via lossy and noisy communication channels resulting in *Networked Control Systems* (NCS). The research in NCS has branched into many different directions that deal with the (individual or combined) effects of delays, limited information exchange, and information losses on the stability of the plant, see, e.g., [8], [9], [15], [23] and the references therein. The designed control laws are not required in general to satisfy a predefined bound and hence may take arbitrarily large values, depending on the state of the system as well as the noise sequence.

Control under information loss in the communication channel has been extensively studied within the Linear Quadratic Gaussian (LQG) framework [13], [19]. Typically, the communication channel(s) are modeled by independent and identically distributed (i.i.d) Bernoulli processes, which assign probabilities to the successful transmission of packets. Perhaps the most well known result in this setting is: When the transmission of sensor and control data packets happens over a network with TCP-like protocols, the closed-loop system under an LQG controller can be mean-square stabilized provided that the probabilities of successful transmission are above a certain threshold. Since the TCP-like protocols enable the receiver to obtain an acknowledgment of whether or not the packets were successfully transmitted, the separation principle holds and the optimal LQG controller is linear in the estimated state. The control inputs in this case are generated via linear state feedback, and hence no hard input bounds may be imposed.

Guaranteeing hard bounds on the control inputs is of paramount importance in applications. Consequently, many researchers have pursued the problem of optimal control and stabilization for linear systems with bounded control inputs, see, for example, [22], [20], [18], [2]. This problem has also received a renewed interest in recent years [17], [21], [4], [10], [7]. In the deterministic setting, it is well-known [24] that global asymptotic stabilization of a linear system $x_{t+1} = Ax_t + Bu_t$ with bounded inputs is possible if and only if the pair (A, B) is stabilizable under unbounded controls and the spectral radius of the system matrix A is at most 1. In the stochastic setting, it was argued in [14] and proven in [6] that ensuring a mean-square bound for every initial condition is not possible for linear systems with bounded control inputs if the system matrix A is unstable. In [17] we established the existence of a policy with sufficiently large control authority that ensures mean-square boundedness of the states of the system under the assumption that A is Lyapunov stable. Although Lyapunov stability of A is a stronger requirement than the spectral radius of A being at most 1, to the best of our knowledge this is the current state of the art.

In general, one would aim for a unifying design framework for NCS that takes into account various kinds of network imperfections, such as delays, limited data rates, quantization, packet dropouts, etc. However, this is extremely difficult and most of the past research effort has been focused on addressing one or at best two of the imperfections. In this article we address the single issue of input channel dropouts and prove that it is possible to stabilize NCS in the sense of bounded variance with bounded control authority, relying on *imperfect state information*. In the article [5], we address the effect of limited data exchange rate, due to quantization, and show that under *full state information* it is possible to obtain bounded state variance with a *finite alphabet*. The setup in [5] is different from the one in this article and the authors are not aware of a method of unifying the two sets of results. Finally, the main result in this technical note generalizes that in [3] from the perfect state information case to the imperfect one.

Notation

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote the conditional expectation given a sub-σ-algebra $\mathcal{F}'$ of $\mathcal{F}$ as $\mathbb{E}_{\mathcal{F}'}[.]$. For any two matrices A and B of compatible dimensions, we denote by $\mathfrak{R}_\kappa(A, B)$ the κ-th step reachability matrix $\mathfrak{R}_\kappa(A, B) \triangleq [\, A^{\kappa-1}B \quad \cdots \quad AB \quad B \,]$. For any matrix M, we let $\sigma_{\min}(M)$ and $\sigma_{\max}(M)$ be its smallest and largest singular values, respectively, and $M^\dagger$ be its Moore-Penrose pseudo-inverse. For a symmetric positive definite matrix P, let $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote its smallest and largest eigenvalues, respectively. We shall denote by $\|\cdot\|$ the usual Euclidean norm and by $\mathbb{N}$ the set of nonnegative integers $\{0, 1, 2, \ldots\}$. In a Euclidean space we denote by $\mathcal{B}_r$ the closed Euclidean ball of radius r centered at the origin. For $r > 0$ let $\mathrm{sat}_r : \mathbb{R}^n \longrightarrow \mathcal{B}_r$ be the radial saturation function, defined by

$$\mathrm{sat}_r(s) = \begin{cases} s & \text{if } s \in \mathcal{B}_r, \\ \frac{rs}{\|s\|} & \text{otherwise.} \end{cases}$$

II. Problem Setup

Consider the following discrete-time stochastic linear system:

$$x_{t+1} = Ax_t + B\tilde{u}_t + w_t \tag{1a}$$

$$y_t = Cx_t + v_t \tag{1b}$$

where $x_t \in \mathbb{R}^n$ is the state, $\tilde{u}_t \in \mathbb{R}^m$ is the control input, $y_t \in \mathbb{R}^p$ is the output, and w_t and v_t are some random process and measurement noise, respectively. We would like to control the system (1) via a communication network, as shown in Fig. 1. The output of the system y_t

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P. Hokayem is with Medium Voltage Drives, ABB Switzerland, Ltd., Turgi 5300, Switzerland (e-mail: peter.al-hokayem@ch.abb.com).

D. Chatterjee is with Systems and Control Engineering, Indian Institute of Technology, Mumbai 400076, India (e-mail: chatterjee@sc.iitb.ac.in).

F. A. Ramponi is with the Department of Information Engineering, Università degli Studi di Brescia, Brescia 25123, Italy (e-mail: federico.ramponi@ing.unibs.it).

J. Lygeros is with the Automatic Control Laboratory, ETH Zürich, Zürich 8092, Switzerland (e-mail: lygeros@control.ee.ethz.ch).

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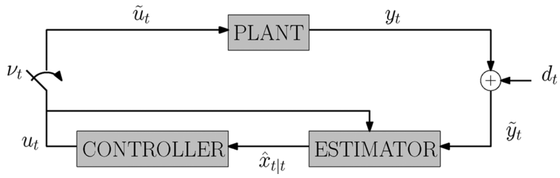

Fig. 1. Networked control system.

is passed through the network and is received at the controller side as $\tilde{y}_t = y_t + d_t$, where d_t is an additional disturbance term introduced by the network, which can model for example the error introduced by a uniform quantizer, with an infinite but countable alphabet, on $\mathbb{R}^p$. At any time t, the estimator utilizes the available information of (σ-algebra generated by) the outputs

$$\mathcal{F}_t = \sigma\{\tilde{y}_0, \dots, \tilde{y}_t\} \tag{2}$$

to generate an estimate $\hat{x}_{t|t}$. This estimate is in turn used by the controller to generate a control input u_t that is transmitted back to the plant via a lossy channel. The channel is characterized by a Bernoulli random variable ν_t that takes values 0 or 1 with probabilities $1-\bar{\nu}$ and $\bar{\nu}$, respectively. Accordingly, the input to the plant is given by $\tilde{u}_t = \nu_t u_t$. We have the following standing assumption.

Assumption 1:

1) The matrix A is orthogonal.
2) The pair (A, B) is reachable in κ steps, i.e., $\operatorname{rank}(\mathfrak{R}_\kappa(A, B)) = n$.
3) The pair (A, C) is observable.
4) The initial condition x_0, the process and measurement noise vectors w_t and v_t, and the input dropout variable ν_t are mutually independent and individually i.i.d., with: $\mathbb{E}\left[\|x_0\|^4\right] \leqslant C_x$, $\mathbb{E}\left[\|w_t\|^4\right] \leqslant C_w$, $\mathbb{E}\left[\|v_t\|^4\right] \leqslant C_v$, and ν_t is a Bernoulli random variable with mean $\bar{\nu}$.
5) The disturbance term d_t satisfies the following uniform bound
$$\|d_t\| \leqslant d_{\max} \quad \forall t \in \mathbb{N}. \tag{3}$$
6) The control inputs are required to satisfy
$$\|u_t\| \leqslant U_{\max} \qquad \forall t \in \mathbb{N}. \tag{4}$$
7) The control inputs $\begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+\kappa-1} \end{bmatrix}$ are generated at times $t = 0, \kappa, 2\kappa, \dots$ and the whole sequence is transmitted as a block across the communication channel at time t.

◇

Assumption 1-1) and 1-2) constitute no loss of generality, since the result may be easily extended to cover the case of A being Lyapunov stable and the pair (A, B) being reachable in κ steps (see the discussion in the Introduction and [17]). Assumption 1-3) is quite standard when dealing with problems with imperfect state information. Assumption 1-4) constitutes a technical requirement similar to those in [17] and [19].

III. Main Result

We shall give a positive answer to the following fundamental question: *Does there exist a causal control policy satisfying* (4), *such that for any initial filtration $\mathcal{F}_0$ the state has bounded variance in closed-loop, i.e., does the following bound*:

$$\sup_{t=0,1,2,\dots} \mathbb{E}_{\mathcal{F}_0}\left[\|x_t\|^2\right] \leqslant \gamma_x \tag{5}$$

hold for some finite constant γ_x?

A. Estimator

Since we do not have access to the full state information at the controller side, we start by designing a state estimator. The prediction step is given by

$$\hat{x}_{t+1|t} \triangleq \mathbb{E}_{\mathcal{F}_t}[x_{t+1}] = A\hat{x}_{t|t} + \bar{\nu}Bu_t \tag{6}$$

with $\hat{x}_{0|-1} = \mathbb{E}[x_0]$, and the update step is given by

$$\begin{aligned} \hat{x}_{t+1|t+1} &\triangleq \mathbb{E}_{\mathcal{F}_{t+1}}[x_{t+1}] = \hat{x}_{t+1|t} + L(\tilde{y}_{t+1} - C\hat{x}_{t+1|t}) \\ &= \hat{x}_{t+1|t} + L(y_{t+1} - C\hat{x}_{t+1|t}) + L(\tilde{y}_{t+1} - y_{t+1}) \\ &= A\hat{x}_{t|t} + \bar{\nu}Bu_t + (\nu_t - \bar{\nu})LCBu_t + LCAe_{t|t} \\ &\quad + LCw_t + Lv_t + Ld_{t+1} \end{aligned} \tag{7}$$

where $e_{t|t} \triangleq x_t - \hat{x}_{t|t}$ is the estimation error and L is a static matrix gain to be chosen so that the matrix $(A - LCA)$ is a Schur stable matrix.

Remark 2: The estimator (6)–(7) has been designed for UDP-like transmission protocols in which there are no acknowledgements of packet reception being sent back from the plant to the estimator (see [19]). It is not the optimal estimator for our problem setup. All the subsequent results also hold with minor changes for the TCP-like protocols, in which the filtration $\mathcal{F}_t$ available to the estimator includes knowledge of the history of the dropout sequence ν_t.

Using (1), (6), and (7), we can see that the estimation (update) error satisfies the following recursion:

$$\begin{aligned} e_{t+1|t+1} = (A - LCA)e_{t|t} &+ (\nu_t - \bar{\nu})(I - LC)Bu_t \\ &+ (I - LC)w_t - Lv_{t+1} - Ld_{t+1}. \end{aligned} \tag{8}$$

The error process in (8) has a bounded conditional fourth moment as shown in the following lemma.

Lemma 3: Let Assumption 1 hold and choose a matrix gain L so that $(A - LCA)$ is Schur stable.[1] Then, there exists a finite positive constant γ_e such that the following bound holds:

$$\sup_{t=0,1,2,\dots} \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|^4\right] \leqslant \gamma_e. \tag{9}$$

Proof: Since $(A - LCA)$ is Schur stable, there exist matrices $Q = Q^T > 0$ and $P = P^T > 0$ that solve the following discrete-time Lyapunov equation:

$$(A - LCA)^T P (A - LCA) - P = -Q. \tag{10}$$

Note that the existence of a stabilizing gain L follows from the facts that (A, C) is observable and the matrix A is orthogonal by Assumption 1.[2] Define $V_t = \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right]$. We have that

$$\begin{aligned} &V_{t+1} - V_t \\ &= \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|e_{t+1|t+1}\right\|_P^4\right] - \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] \\ &= \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A - LCA)e_{t|t} + T_t\right\|_P^4\right] - \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] \end{aligned} \tag{11}$$

[1]It is always possible to choose such a matrix L, as will be seen in the proof.

[2]Since the pair (A, C) is observable, there exits a matrix K such that $(A - KC)$ is Schur stable or equivalently, there exists symmetric positive definite matrices $\tilde{P}$ and $\tilde{Q}$ satisfying the Lyapunov equation $(A - KC)^T\tilde{P}(A - KC) - \tilde{P} = -\tilde{Q}$. Let $L = A^T K$, see [1, Proposition 11.10.5]. Then we have that $(A - ALC)^T\tilde{P}(A - ALC) - \tilde{P} = -\tilde{Q}$, or equivalently, by the orthogonality of A, $A(A - LCA)^T A^T \tilde{P} A (A - LCA) A^T - \tilde{P} = -\tilde{Q}$. Multiplying the last equality by A^T and A from the left and right, respectively, and setting $P = A^T\tilde{P}A$ and $Q = A^T\tilde{Q}A$ yields the result.

where $T_t = (\nu_t - \bar{\nu})(I-LC)Bu_t + (I-LC)w_t - Lv_{t+1} - Ld_{t+1}$. Expanding the first term in (11) and upper-bounding the result, we obtain

$$V_{t+1} - V_t \leqslant \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P^4\right] - \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] + S_t \quad (12)$$

where

$$\begin{aligned} S_t = & 4\mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P^3 \|T_t\|_P\right] \\ & + 6\mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P^2 \|T_t\|_P^2\right] \\ & + 4\mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P \|T_t\|_P^3\right] \\ & + \mathbb{E}_{\mathcal{F}_{t+1}}\left[\|T_t\|_P^4\right]. \end{aligned} \quad (13)$$

Moreover, we have that

$$\begin{aligned} \|T_t\|_P \leqslant T_t' \triangleq & \|(I-LC)B\|_P U_{\max} \\ & + \|(I-LC)w_t - Lv_{t+1}\|_P + \|L\|_P d_{\max} \end{aligned} \quad (14)$$

where we have used the bounds (3) and (4) and the fact that $|\nu_t - \bar{\nu}| \leqslant 1$. Using the upper bound (14), the independence of $e_{t|t}$, w_t and v_{t+1}, and Jensen's inequality,[3] we can further upper-bound S_t as follows:

$$S_t \leqslant c_0 + c_1 V_t^{1/4} + c_2 V_t^{1/2} + c_3 V_t^{3/4} \quad (15)$$

with the constants

$$\begin{aligned} c_0 &= \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|T_t'\right\|_P^4\right], \\ c_1 &= 4\|A-LCA\|_P \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|T_t'\right\|_P^3\right], \\ c_2 &= 6\|A-LCA\|_P^2 \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|T_t'\right\|_P^2\right], \end{aligned}$$

and

$$c_3 = 4\|A-LCA\|_P^3 \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|T_t'\right\|_P\right.$$

The constants c_0 through c_3 may be computed using the upper bound (14) and the statistics of w_t and v_t. Returning to (12), we have that

$$\begin{aligned} & \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P^4\right] - \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] \\ & = \mathbb{E}_{\mathcal{F}_t}\left[\left(\left\|(A-LCA)e_{t|t}\right\|_P^2 - \left\|e_{t|t}\right\|_P^2\right)\right. \\ & \quad \left.\times\left(\left\|(A-LCA)e_{t|t}\right\|_P^2 + \left\|e_{t|t}\right\|_P^2\right)\right] \end{aligned}$$

since $e_{t|t}$ is independent of the measurement $\tilde{y}_{t+1}$. It follows from (10) that

$$\begin{aligned} & \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left\|(A-LCA)e_{t|t}\right\|_P^4\right] - \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] \\ & = \mathbb{E}_{\mathcal{F}_t}\left[-\left\|e_{t|t}\right\|_Q^2\left(\left\|(A-LCA)e_{t|t}\right\|_P^2 + \left\|e_{t|t}\right\|_P^2\right)\right] \\ & \leqslant \mathbb{E}_{\mathcal{F}_t}\left[-\left\|e_{t|t}\right\|_Q^2\left\|e_{t|t}\right\|_P^2\right] \\ & \leqslant -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|_P^4\right] \end{aligned} \quad (16)$$

[3]For $i = 1, 2, 3$ we have that

$$\mathbb{E}_{\mathcal{F}_{t+1}}\left[\|e_{t|t}\|_P^i\right] = \mathbb{E}_{\mathcal{F}_{t+1}}\left[\left(\|e_{t|t}\|_P^4\right)^{i/4}\right] \leqslant \mathbb{E}_{\mathcal{F}_{t+1}}\left[\|e_{t|t}\|_P^4\right]^{i/4}$$

since $\varphi(\cdot) \triangleq (\cdot)^{i/4}$ is a concave function.

Combining (12), (15), and (16), we obtain

$$V_{t+1} - V_t \leqslant -\lambda V_t + c_0 + c_1 V_t^{1/4} + c_2 V_t^{1/2} + c_3 V_t^{3/4}$$

where $\lambda \triangleq \lambda_{\min}(Q)/\lambda_{\max}(P) \leqslant 1$.[4] Hence, for a sufficiently large V_t, for example

$$V_t > c = \max\left\{\frac{8c_0}{\lambda}, \left(\frac{8c_1}{\lambda}\right)^{4/3}, \left(\frac{8c_2}{\lambda}\right)^2, \left(\frac{8c_3}{\lambda}\right)^4\right\}$$

we have that

$$V_{t+1} \leqslant \left(1 - \frac{\lambda}{2}\right) V_t.$$

Iterating the last inequality, we obtain the following bound:

$$V_t \leqslant \left(1-\frac{\lambda}{2}\right)^t V_0 + \sum_{i=0}^{t-1}\left(1-\frac{\lambda}{2}\right)^i c \leqslant V_0 + \frac{2c}{\lambda} < \infty.$$

Therefore

$$\begin{aligned} & \sup_{t=0,1,2,\ldots} \mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|^4\right] \\ & \leqslant \sup_{t=0,1,2,\ldots} \frac{1}{\lambda_{\min}(P)^2} V_t \\ & \leqslant \frac{1}{\lambda_{\min}(P)^2}\left(V_0 + \frac{2c}{\lambda}\right) \\ & \leqslant \frac{1}{\lambda_{\min}(P)^2}\left(\lambda_{\max}(P)^2 \mathbb{E}_{\mathcal{F}_0}\left[\left\|e_{0|0}\right\|^4\right] + \frac{2c}{\lambda}\right) \triangleq \gamma_e. \end{aligned}$$

Finally, $\mathbb{E}_{\mathcal{F}_0}\left[\left\|e_{0|0}\right\|^4\right]$ is bounded by Assumption 1-4) on x_0, (1b), and the definition of $\hat{x}_{0|-1}$ in (6). ■

We are now ready to state our main result.

Theorem 4: Let Assumption 1 hold and choose the matrix L as in Lemma 3. Then, there exist an average dropout threshold ν^* and a causal deterministic κ-step control policy satisfying (4) with a minimal control authority $U_{\max}$ (depending on ν^* and the problem parameters), such that for all $\bar{\nu} \in (\nu^*, 1]$ the closed-loop system is mean-square bounded.

Proof: The state variance in (5) may be upper-bounded as follows:

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_0}\left[\|x_t\|^2\right] &= \mathbb{E}_{\mathcal{F}_0}\left[\left\|\hat{x}_t + e_{t|t}\right\|^2\right] \\ &\leqslant 2\mathbb{E}_{\mathcal{F}_0}\left[\|\hat{x}_t\|^2\right] + 2\mathbb{E}_{\mathcal{F}_0}\left[\left\|e_{t|t}\right\|^2\right]. \end{aligned} \quad (17)$$

The error variance in (17) is bounded by Lemma 3, since

$$\mathbb{E}_{\mathcal{F}_0}\left[\left\|e_{t|t}\right\|^2\right] = \mathbb{E}_{\mathcal{F}_0}\left[\mathbb{E}_{\mathcal{F}_t}\left[\left\|e_{t|t}\right\|^2\right]\right] \leqslant \sqrt{\gamma_e} \quad (18)$$

where the last bound follows from Jensen's inequality. Therefore, we only need to show that there exists a causal deterministic κ-step causal control policy satisfying (4) that ensures that $\sup_{t=0,1,2,\ldots} \mathbb{E}_{\mathcal{F}_0}\left[\|\hat{x}_t\|^2\right]$

[4]It holds

$$\begin{aligned} & \lambda_{\min}(Q)\|x\|^2 \\ & \leqslant x^T Q x = x^T(P - (A-LCA)^T P(A-LCA))x \\ & \leqslant x^T P x \leqslant \lambda_{\max}(P)\|x\|^2 \end{aligned}$$

is finite. This is achieved by considering the sub-sampled process $(\hat{x}_{t|t})_{t=0,\kappa,2\kappa,\ldots}$, given by

$$\begin{aligned}
&\hat{x}_{t+\kappa|t+\kappa} \\
&= A^{\kappa}\hat{x}_{t|t} + \bar{\nu}\mathfrak{R}_{\kappa}(A,B)\begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1}\end{bmatrix} \\
&\quad + (\nu_t - \bar{\nu})\mathfrak{R}_{\kappa}(A,LCB)\begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1}\end{bmatrix} \\
&\quad + \mathfrak{R}_{\kappa}(A,LCA)\begin{bmatrix} e_{t|t} \\ \vdots \\ e_{t+\kappa-1|t+\kappa-1}\end{bmatrix} + \mathfrak{R}_{\kappa}(A,LC)\begin{bmatrix} w_t \\ \vdots \\ w_{t+\kappa-1}\end{bmatrix} \\
&\quad + \mathfrak{R}_{\kappa}(A,L)\begin{bmatrix} v_{t+1} \\ \vdots \\ v_{t+\kappa}\end{bmatrix} + \mathfrak{R}_{\kappa}(A,L)\begin{bmatrix} d_{t+1} \\ \vdots \\ d_{t+\kappa}\end{bmatrix}
\end{aligned} \tag{19}$$

and showing that it is mean-square bounded, from which the required result follows.

Our proof that the sub-sampled process (19) is mean-square bounded relies on the following (immediate) adaptation of the fundamental result [16, Theorem 1].

Proposition 5: Let $(\xi_t)_{t\in\mathbb{N}}$ be a sequence of nonnegative random variables on some probability space $(\Omega, \mathcal{F}_t, \mathsf{P})$, and let $(\mathcal{F}_t)_{t\in\mathbb{N}}$ be any filtration to which $(\xi_t)_{t\in\mathbb{N}}$ is adapted. Suppose that there exist constants $b > 0$, and $J, M < \infty$, such that $\xi_0 \leqslant J$, and for all t

$$\mathbb{E}_{\mathcal{F}_t}\Big[\xi_{t+1} - \xi_t\Big] \leqslant -b \quad \text{on the event } \{\xi_t > J\}, \text{ and} \tag{20}$$

$$\mathbb{E}_{\{\xi_0,\ldots,\xi_t\}}\Big[(\xi_{t+1} - \xi_t)^4\Big] \leqslant M. \tag{21}$$

Then there exists a constant $\gamma = \gamma(b, J, M) > 0$ such that $\sup_{t\in\mathbb{N}} \mathbb{E}\Big[\xi_t^2\Big] \leqslant \gamma$. □

Henceforth, let $\mathcal{F}_t$ be as in (2). Accordingly, we have that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_t}\left[\left\|\hat{x}_{t+\kappa|t+\kappa}\right\| - \left\|\hat{x}_{t|t}\right\|\right] \\
&\leqslant \mathbb{E}_{\mathcal{F}_t}\left[\left\|A^{\kappa}\hat{x}_{t|t} + \bar{\nu}\mathfrak{R}_{\kappa}(A,B)\begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1}\end{bmatrix}\right\| - \left\|\hat{x}_{t|t}\right\|\right] \\
&\quad + \mathbb{E}_{\mathcal{F}_t}\left[\left\|(\nu_t - \bar{\nu})\mathfrak{R}_{\kappa}(A,LCB)\begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1}\end{bmatrix}\right\|\right] \\
&\quad + \mathbb{E}_{\mathcal{F}_t}\left[\left\|\mathfrak{R}_{\kappa}(A,LCA)\begin{bmatrix} e_{t|t} \\ \vdots \\ e_{t+\kappa-1|t+\kappa-1}\end{bmatrix}\right\|\right] \\
&\quad + \mathbb{E}_{\mathcal{F}_t}\left[\left\|\mathfrak{R}_{\kappa}(A,LC)\begin{bmatrix} w_t \\ \vdots \\ w_{t+\kappa-1}\end{bmatrix}\right\|\right] \\
&\quad + \mathbb{E}_{\mathcal{F}_t}\left[\left\|\mathfrak{R}_{\kappa}(A,L)\begin{bmatrix} v_{t+1} \\ \vdots \\ v_{t+\kappa}\end{bmatrix}\right\|\right] \\
&\quad + \mathbb{E}_{\mathcal{F}_t}\left[\left\|\mathfrak{R}_{\kappa}(A,L)\begin{bmatrix} d_{t+1} \\ \vdots \\ d_{t+\kappa}\end{bmatrix}\right\|\right].
\end{aligned} \tag{22}$$

By Lemma 3 and Assumption 1 there exists a constant c given by

$$\begin{aligned}
c = \sqrt{\kappa}\Big(&\|\mathfrak{R}_{\kappa}(A,LCA)\| \sqrt[4]{\gamma_e} + \|\mathfrak{R}_{\kappa}(A,LC)\| \sqrt[4]{C_w} \\
&+ \|\mathfrak{R}_{\kappa}(A,L)\| \sqrt[4]{C_v} + \|\mathfrak{R}_{\kappa}(A,L)\| d_{\max}\Big)
\end{aligned} \tag{23}$$

that upper-bounds the last four terms in (22). In the spirit of [11], we propose the following control policy

$$U_t \triangleq \begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1}\end{bmatrix} \triangleq -\mathfrak{R}_{\kappa}(A,B)^{\dagger}\operatorname{sat}_r(A^{\kappa}\hat{x}_{t|t}) \tag{24}$$

for some radius $r > 0$ of the saturation function to be specified. The policy (24) is generated every κ steps using the state estimate $\hat{x}_{t|t}$ and is sent to the plant through the lossy network according to Assumption 1-7). As such, the sequence of inputs $\{u_t, \ldots, u_{t+\kappa-1}\}$ is either received by the plant if $\nu_t = 1$ or it is dropped if $\nu_t = 0$. Plugging the policy (24) into (22) and using the constant c in (23), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_t}\left[\left\|\hat{x}_{t+\kappa|t+\kappa}\right\| - \left\|\hat{x}_{t|t}\right\|\right] \\
&\leqslant -\bar{\nu}r + 2\bar{\nu}(1-\bar{\nu})\left\|\mathfrak{R}_{\kappa}(A,LCB)\mathfrak{R}_{\kappa}(A,B)^{\dagger}\right\| r + c \\
&= -\bar{\nu}r\left(1 - 2(1-\bar{\nu})\left\|\mathfrak{R}_{\kappa}(A,LCB)\mathfrak{R}_{\kappa}(A,B)^{\dagger}\right\|\right) + c.
\end{aligned}$$

Pick an $\epsilon > 0$ and define the dropout threshold $\nu^* \triangleq 1 - ((1-\epsilon)/2\left\|\mathfrak{R}_{\kappa}(A,LCB)\mathfrak{R}_{\kappa}(A,B)^{\dagger}\right\|)$. Then, $\forall \bar{\nu} \in (\nu^*, 1]$

$$\mathbb{E}_{\mathcal{F}_t}\left[\left\|\hat{x}_{t+\kappa|t+\kappa}\right\| - \left\|\hat{x}_{t|t}\right\|\right] \leqslant -\bar{\nu}r\epsilon + c \leqslant -\nu^* r\epsilon + c.$$

For any $b > 0$, we have that

$$-\nu^* r\epsilon + c \leqslant -b \tag{25}$$

on the event $\left\{\left\|\hat{x}_{t|t}\right\| \geqslant J \triangleq \max\left\{\left\|\hat{x}_{0|0}\right\|, r\right\}\right\}$, if we choose the radius $r \geqslant b + c/\nu^*\epsilon$. Accordingly, using (24), the required control authority is defined as $U_{\max} \triangleq \sigma_{\max}\left(\mathfrak{R}_{\kappa}(A,B)^{\dagger}\right) \cdot r$, and condition (20) of Proposition 5 is satisfied for $\xi_t \triangleq \left\|\hat{x}_{t|t}\right\|$.

We proceed to show that the condition (21) is satisfied as well. The following upper bound holds:

$$\begin{aligned}
&\mathbb{E}_{\{\|\hat{x}_{0|0}\|,\ldots,\|\hat{x}_{t|t}\|\}}\left[\left(\left\|\hat{x}_{t+\kappa|t+\kappa}\right\| - \left\|\hat{x}_{t|t}\right\|\right)^4\right] \\
&\leqslant \mathbb{E}_{\{\|\hat{x}_{0|0}\|,\ldots,\|\hat{x}_{t|t}\|\}}\Bigg[\Bigg(-r\bar{\nu} \\
&\quad + (\nu_t - \bar{\nu})\left\|\mathfrak{R}_{\kappa}(A,LCB)\mathfrak{R}_{\kappa}(A,B)^{\dagger}\right\| r \\
&\quad + \|\mathfrak{R}_{\kappa}(A,LCA)\|\left\|\begin{bmatrix} e_{t|t} \\ \vdots \\ e_{t+\kappa-1|t+\kappa-1}\end{bmatrix}\right\| \\
&\quad + \|\mathfrak{R}_{\kappa}(A,LC)\|\left\|\begin{bmatrix} w_t \\ \vdots \\ w_{t+\kappa-1}\end{bmatrix}\right\| \\
&\quad + \|\mathfrak{R}_{\kappa}(A,L)\|\left\|\begin{bmatrix} v_{t+1} \\ \vdots \\ v_{t+\kappa}\end{bmatrix}\right\| + \mathfrak{R}_{\kappa}(A,L)\sqrt{\kappa}d_{\max}\Bigg)^4\Bigg]
\end{aligned} \tag{26}$$

where we have used the policy (24), the fact that $\left\|\hat{x}_{t|t}\right\| = \left\|A^{\kappa}\hat{x}_{t|t}\right\|$ since A is orthogonal, and the upper bound (3) on the channel-induced disturbance d_t. It follows from Lemma 3 and Assumption 1-4) that there exists a bound M, such that

$$\mathbb{E}_{\{\|\hat{x}_{0|0}\|,\ldots,\|\hat{x}_{t|t}\|\}}\left[\left(\left\|\hat{x}_{t+\kappa|t+\kappa}\right\| - \left\|\hat{x}_{t|t}\right\|\right)^4\right] \leqslant M \tag{27}$$

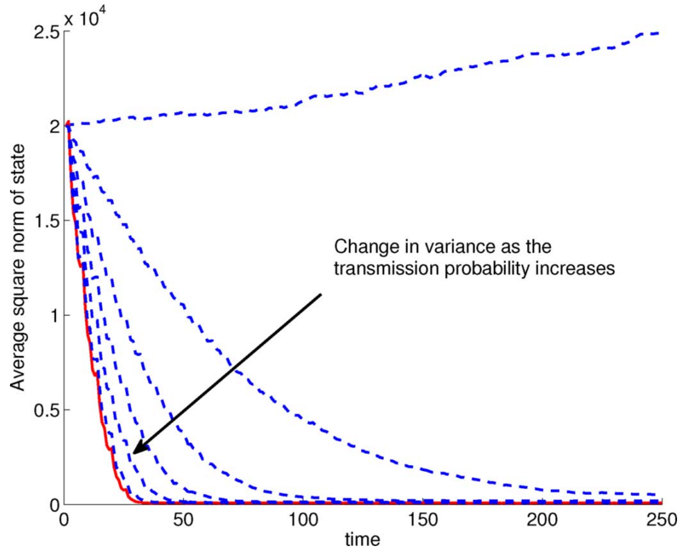

Fig. 2. Empirical variance of the state with 500 sample paths of the noise sequences $(w_t)_{t=0,1,2,\ldots}$ and $(v_t)_{t=0,1,2,\ldots}$ and input drop sequences $(\nu_t)_{t=0,\kappa,2\kappa,\ldots}$: The results for for various choices of transmission probability $\bar{\nu} = 0, 0.2, 0.4, \ldots, 0.8, 1$ with state estimation are shown in dashed blue lines, while the results for the the policy in [17] with full state information is shown in the solid red line.

and condition (21) is satisfied for $\xi_t \triangleq \left\|\hat{x}_{t|t}\right\|$. Therefore, by Proposition 5 there exists a constant $\gamma'_{\hat{x}}$ such that

$$\sup_{t=0,\kappa,2\kappa,\ldots} \mathbb{E}_{\mathcal{F}_0}\left[\left\|\hat{x}_{t|t}\right\|^2\right] \leqslant \gamma'_{\hat{x}}. \tag{28}$$

By the linearity of the dynamics (7), the boundedness of the control inputs, Lemma 3, and Assumption 1-4), we can show using a similar argument as in [17] the existence of a constant $\gamma_{\hat{x}}$ depending on $\gamma'_{\hat{x}}$ such that

$$\sup_{t=0,1,2,\ldots} \mathbb{E}_{\mathcal{F}_0}\left[\left\|\hat{x}_{t|t}\right\|^2\right] \leqslant \gamma_{\hat{x}}. \tag{29}$$

Finally, setting $\gamma_x \triangleq 2\gamma_e + 2\gamma_{\hat{x}}$ completes the proof. ■

IV. Example

Consider the system

$$x_{t+1} = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t + w_t$$
$$y_t = x_t + v_t$$

with $w_t \sim \mathcal{N}(0, 10I) v_t \sim \mathcal{N}(0, 2)$. We simulated the policy proposed in this technical note with $d_{\max} = 0$, $\hat{x}_{0|-1} = 0$, $U_{\max} = 12.25$ starting from the initial condition $x_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ and an initial estimate $\hat{x}_{0|-1} = 0$ for 500 sample paths of the noise sequences w_t and v_t and drop sequences ν_t. The matrix L used in the estimator was chosen to be $L = \begin{bmatrix} 0.8252 & -0.4400 \\ 0.3028 & 0.7460 \end{bmatrix}$, which places the poles of $(A - LCA)$ at 0.3497 and 0.5081.

Fig. 2 depicts the results for the empirical average achieved for various choices of the average drop rate $\bar{\nu}$ in $[0, 1]$. The variance of the state grows unbounded for a choice of $\bar{\nu} = 0$ and as $\bar{\nu}$ increases towards 1, the variance of the state approaches (in a uniform way) the one achieved utilizing the policy proposed in [17] with full state feedback. The difference between the state variance achieved by using the policy proposed in this technical note for $\bar{\nu} = 1$ versus the one achieved by using the one proposed in [17] is due to the incurred estimation error. Finally, note that the chosen $U_{\max}$ is lower than the one required for obtaining the theoretical mean-square bounds on the state in Theorem 4 and that our control policy is still stabilizing for a value of $\bar{\nu} = 0.2$ that is much lower than the required $\nu^* = 1 - ((1-\epsilon)/2 \left\|\mathfrak{R}_\kappa(A, LCB)\mathfrak{R}_\kappa(A, B)^\dagger\right\|)$. In other words, our main result establishes conditions which are sufficient, but clearly conservative.

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