On Mean-Square Boundedness of Stochastic Linear Systems With Quantized Observations

Debasish Chatterjee, Peter Hokayem, Federico A. Ramponi, and John Lygeros

***Abstract*—We propose a procedure to design a state-quantizer with a *fixed finite alphabet* for a Lyapunov stable stochastic linear system, and a bounded policy based on the resulting quantized state measurements to ensure bounded second moments of the states in closed-loop.**

***Index Terms*—Static quantizer, transmission rate.**

I. Introduction

Recently the authors have investigated the stabilizability in the mean-square sense of a controlled discrete-time linear system subject to unbounded random disturbance. Namely, given a linear system $x_{t+1} = Ax_t + Bu_t + w_t$, where (A, B) is a stabilizable pair, $(u_t)_{t\in\mathbb{N}_0}$ is a bounded control signal ($\|u_t\| \leqslant U_{\max}$ for all t), and $(w_t)_{t\in\mathbb{N}_0}$ is a sequence of independent random vectors, we have demonstrated that with an appropriate choice of the control strategy, it is possible to attain mean-square boundedness of the state process (i.e., $\sup_{t\geqslant 0} \mathsf{E}[\|x_t\|^2] \leqslant \infty$) provided that $(w_t)_{t\in\mathbb{N}_0}$ has bounded fourth moment.[1] The reader is referred to our articles [13] and [2], which deal with the case when full-state information is available; the more recent article [7] treats this problem in greater generality and establishes the result by means of a receding-horizon strategy.

It is of interest to extend the preceding result to the context of networked systems, i.e., supposing that the signals involved in both the state measurement and the actuation of the control action travel across a communication network. At least two key problems arise: first, all the information must be encoded, and in particular the state information is quantized; second, at certain times either the state information or the control action can be absent due to communication packet drops. The aim of this article is to deal with the first problem: we assume that the state information reaches the controller in a quantized form, and we propose a controller that ensures that the states of a Lyapunov stable system are mean-square bounded.[2]

Stabilization of linear systems with quantized state measurements has a rich history, see, for example, [1], [3]–[5], [8]–[11], [14]–[17]. While this article deals neither with networked control in its broad generality, nor with the case of multiple systems interacting over a network per se, it provides a contribution to the subject by proving that ensuring mean-square stability under unbounded noise and bounded control actions—a highly nontrivial task for Lyapunov stable linear systems even with full-state information—is possible with quantized state information. We stress that, unlike elsewhere in the literature, we do not focus on attaining stability under the assumption of a finite *transmission rate* available between sensors, controller, and actuators; our main result is not in terms of a minimum necessary rate. We establish, instead, that mean-square boundedness can be attained with the information provided by a *fixed* quantizer with *finite alphabet*—that is, knowing in which region of the state space, or "bin," out of finitely many, the state happens to lie at prescribed times—and our main result is in terms of the maximum "amplitude" of such bins.

The proof of our result is constructive: we show that the information relevant to ensure mean-square boundedness is encoded in the *direction* of the state vector. We therefore start by constructing a finite partition (the "bins") of the set of all possible directions, and a corresponding quantizer. Based on this fixed quantizer with finitely many values, we then define a time-varying policy as a concatenation of a κ-length policy $(u_{\kappa t}, u_{\kappa t+1}, \cdots, u_{(\kappa+1)t-1})$, that depends only on the "bin" in which the state happens to fall at times κt.

As a consequence, the control actions can take only a finite number of values as well; hence, mean-square stability is trivially attained also with bounded transmission rate. Under mild assumptions on the control bound $U_{\max}$ and on the maximum size of the bins, (and as a consequence their number,) this policy ensures mean-square boundedness of the states.

In the same vein, note that on the one hand, for linear systems with at least one eigenvalue outside the closed unit disc, it is impossible to attain mean-square boundedness of the states with bounded control actions [2]. Observe that although the finite data rate theorem [9, Theorem 2.1] asserts that it is possible to bound the variance of a stochastic linear system with unbounded noise by appropriate controls transmitted over a channel supporting a bounded data rate, the *magnitude* of the control needed must necessarily be unbounded if the system matrix is unstable. This was argued following Proposition 5.1 in [9] and established in [2], and the technical problems involved are quite different from those concerned with stabilization of deterministic systems with quantized states, considered e.g., in [11], where the disturbances are bounded. Due to the finiteness of the input set, the result of this article cannot be extended to unstable systems.

On the other hand, for asymptotically stable systems, *any* bounded sequence of controls is enough to ensure mean-square bounded states; therefore, a trivial quantizer is good enough. For Lyapunov stable systems that are not asymptotically stable, however, not all standard quantizers may work.

To summarize, our result shows that for the (borderline) case of Lyapunov stable systems, a specifically designed *static quantizer with*

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D. Chatterjee is with the Systems and Control Engineering, Indian Institute of Technology-Bombay, Powai, Mumbai-400076, Maharashtra, India (e-mail: dchatter@iitb.ac.in).

P. Hokayem is with Medium Voltage Drives, ABB Switzerland Ltd., Turgi 5300, Switzerland (e-mail: peter.al-hokayem@ch.abb.com).

F. A. Ramponi is with the Department of Information Engineering, Università degli Studi di Brescia, Brescia 25123, Italy (e-mail: federico.ramponi@ing.unibs.it).

J. Lygeros is with the Automatic Control Laboratory, ETH Zürich, Zürich 8092, Switzerland (e-mail: lygeros@control.ee.ethz.ch).

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[1]Observe that this fourth moment requirement is more general than the stipulation that $(w_t)_{t\in\mathbb{N}_0}$ is itself bounded, which is the standard assumption in robust control.

[2]The second problem can also be solved assuming a *linear* quantizer—treating the "quantization noise" as a further bounded disturbance; for a solution to this problem, we refer the reader to the companion article [6], which provides a standard stochastic model for the packet drops, imperfect and partial state observation, and bounded control actions.

finitely many fixed bins suffices, and, as such, fills the gap between the two extreme cases above. The result is tested on a simple system at the end of the article.

II. The Result

Consider the linear control system

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 \text{ given}, \quad t = 0, 1, \ldots \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the vector of states, $u_t \in \mathbb{R}^m$ is the vector of control actions, $(w_t)_{t\in\mathbb{N}_0}$ is a zero-mean sequence of noise vectors defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and A and B are matrices of appropriate dimensions. It is assumed that instead of perfect measurements of the state, quantized state measurements are available by means of a quantizer $\mathbf{q} : \mathbb{R}^n \longrightarrow Q$, where $Q \subset \mathbb{R}^n$ is a finite set of vectors in $\mathbb{R}^n$ which we refer to as "bins".

Our objective is to construct a quantizer and a corresponding control policy such that the magnitude of the control is *uniformly bounded*, (i.e., for some $U_{\max} > 0$ we have $\|u_t\| \leqslant U_{\max}$ for all t,) the number of bins Q is *finite*, and the state of (1) is *mean-square bounded* (i.e., $\sup_{t\in\mathbb{N}_0} \mathsf{E}_{x_0}[\|x_t\|^2] < \infty$,) in closed-loop.

Assumption 1:

- The matrix A is Lyapunov stable—the eigenvalues of A have magnitude at most 1, and those on the unit circle have equal geometric and algebraic multiplicities.
- The pair (A, B) is reachable in κ steps, i.e., $\operatorname{rank}(B\ AB \cdots A^{\kappa-1}B) = n$.
- $(w_t)_{t\in\mathbb{N}_0}$ is a zero mean sequence of independent vectors with $C_4 := \sup_{t\in\mathbb{N}_0} \mathsf{E}[\|w_t\|^4] < \infty$.
- $\|u_t\| \leqslant U_{\max}$ for all $t \in \mathbb{N}_0$.

The policy that we construct below belongs to the class of κ-history-dependent policies, where "κ-history" refers to the κ quantized states preceding the current time step. We refer the reader to [13] for the basic setup, definitions, and in particular to [13, §3.4] for details about a change of basis in $\mathbb{R}^n$ that shows that it is sufficient to consider A orthogonal. We let $\mathcal{R}_k(A, M) := (A^{k-1}M \cdots AM\ M)$ for a matrix M of appropriate dimensions, let $M^\dagger$ denote the Moore–Penrose pseudoinverse of M, and let $\sigma_{\min}(M), \sigma_{\max}(M)$ denote the minimal and maximal singular values of M, respectively. I denotes the $d \times d$ identity matrix. For a $0 \neq v \in \mathbb{R}^n$, let $\Pi_v(\cdot) := \langle \cdot, (v/\|v\|)\rangle(v/\|v\|)$ and $\Pi_v^\perp(\cdot) := I - \Pi_v(\cdot)$ denote the projections onto the span of v and its orthogonal complement, respectively. For $r > 0$ let the radial r-saturation function $\operatorname{sat}_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined as $\operatorname{sat}_r(y) := \min\{r, \|y\|\}(y/\|y\|)$, and let $B_r \subset \mathbb{R}^n$ denote the open r ball centered at 0 and ∂B_r denote its boundary.

Theorem 2: Consider the system (1), and suppose that Assumption 1 holds. Let the quantizer be such that there exists a constant r satisfying:

a) $r > (\sqrt{\kappa}\sigma_{\max}(\mathcal{R}_\kappa(A, I))\sqrt[4]{C_4}/\cos(\varphi) - \sin(\varphi))$, *where* $\varphi \in [0, \pi/4[$ *is the maximal angle between* z *and* $\mathbf{q}(z)$, $z \notin B_r$, *and*

b) $\mathbf{q}(z) = \mathbf{q}(\operatorname{sat}_r(z)) \in \partial B_r$ for every $z \notin B_r$.

Let $U_{\max} \geqslant r/\sigma_{\min}(\mathcal{R}_\kappa(A, B))$. *Then successive* κ*-step applications of the control policy*

$$\left(u_{\kappa t}^{\mathsf{T}} \cdots u_{\kappa(t+1)-1}^{\mathsf{T}}\right)^{\mathsf{T}} := -\mathcal{R}_\kappa(A, B)^\dagger A^\kappa \mathbf{q}(x_{\kappa t}), \quad t \in \mathbb{N}_0 \tag{2}$$

ensures that $\sup_{t\in\mathbb{N}_0} \mathsf{E}_{x_0}[\|x_t\|^2] < \infty$.

Remark 3: Observe that Theorem 2 outlines a procedure for constructing a quantizer with finitely many bins, an example of which on $\mathbb{R}^2$ is depicted in Fig. 1. We see from the hypotheses of Theorem 2 that the quantizer must be radial, and have no large gap between the bins on the r-sphere; the quantization rule for the states inside B_r does not matter insofar as mean-square boundedness of the states is concerned. As a consequence of the control policy in Theorem 2, the control alphabet is also finite with $\kappa|Q|$ elements. Since the number of

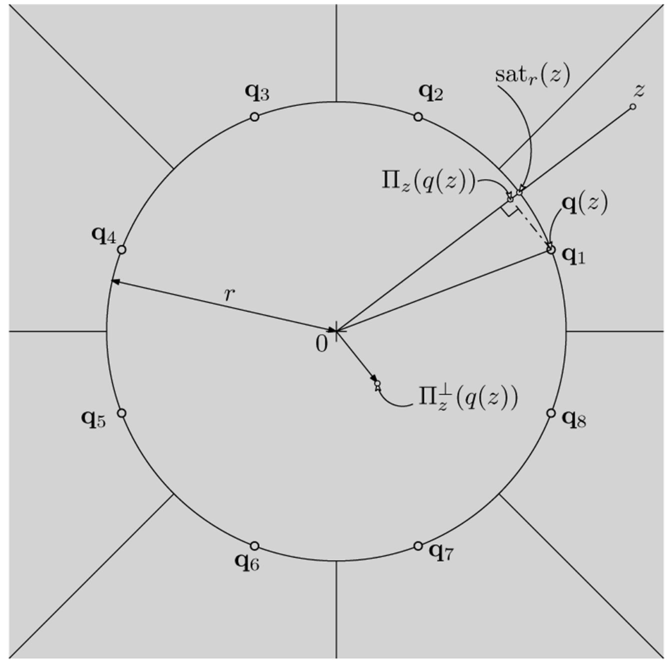

Fig. 1. Pictorial depiction of the proposed quantization scheme in $\mathbb{R}^2$, with $\{\mathbf{q}_0 = 0, \mathbf{q}_1, \ldots, \mathbf{q}_8\}$ being the set of bins. The various projections are computed for a generic state z outside the r-ball centered at the origin.

orthants grows exponentially with n (the dimension of x) and since $\varphi \in [0, \pi/4[$, the number of bins also increases at least exponentially with n. Moreover, for quantizers $\mathbf{q}$ such that

- $\varphi \searrow 0$, i.e., the "density" of the bins on the r-sphere increases indefinitely due to $(\cos\varphi - \sin\varphi) \xrightarrow{\varphi\searrow 0} 1$, and
- $\|z - \mathbf{q}(z)\| \searrow 0$ uniformly in B_r,

(2) tends uniformly to the policy proposed in [13]. ◃

Hereafter $\mathsf{E}^{\mathcal{F}'}[\cdot]$ denotes conditional expectation for a σ-algebra $\mathcal{F}' \subset \mathcal{F}$. We need the following immediate consequence of [12, Theorem 1].

Proposition 4: Let $(\xi_t)_{t\in\mathbb{N}_0}$ *be a sequence of nonnegative random variables on some probability space* $(\Omega, \mathcal{F}, \mathsf{P})$, *and let* $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ *be any filtration to which* $(\xi_t)_{t\in\mathbb{N}_0}$ *is adapted. Suppose that there exist constants* $b > 0$, *and* $J, M < \infty$, *such that* $\xi_0 \leqslant J$, *and for all* t *we have* $\mathsf{E}^{\mathcal{F}_t}[\xi_{t+1} - \xi_t] \leqslant -b$ *on the event* $\{\xi_t > J\}$ *and* $\mathsf{E}[|\xi_{t+1} - \xi_t|^4|\xi_0, \ldots, \xi_t] \leqslant M$. *Then there exists a constant* $\gamma = \gamma(b, J, M) > 0$ *such that* $\sup_{t\in\mathbb{N}_0} \mathsf{E}[\xi_t^2] \leqslant \gamma$.

Preparatory to the proof of Theorem 2, observe that it is no loss of generality to assume that the matrix A is orthogonal. Indeed, as argued in [13, Proof of Theorem 1], there exists a change of basis which decomposes a given system matrix A into its real Jordan form, and after a rearrangement it is possible to block-diagonalize A into an asymptotically stable part and an orthogonal part, which yield two decoupled subsystems. While mean-square boundedness of the former subsystem under bounded controls follows at once from standard Lyapunov techniques, the latter requires further analysis, and accordingly we shall henceforth consider A to be orthogonal.

Proof of Theorem 2: Let $\mathcal{F}_t$ be the σ-algebra generated by $\{x_s | s = 0, \ldots, t\}$. Since $\mathbf{q}$ is a measurable map, it is clear that $(\mathbf{q}(x_t))_{t\in\mathbb{N}_0}$ is $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$-adapted. For $t \in \mathbb{N}_0$, on $\{\|x_{\kappa t}\| > r\}$

$$\begin{aligned}\mathsf{E}^{\mathcal{F}_{\kappa t}}&\left[\|x_{\kappa(t+1)}\| - \|x_{\kappa t}\|\right] \\ &= \mathsf{E}^{\mathcal{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A, B)\bar{u}_{\kappa t} + \bar{w}_{\kappa t}\| - \|x_{\kappa t}\|\right]\end{aligned}$$

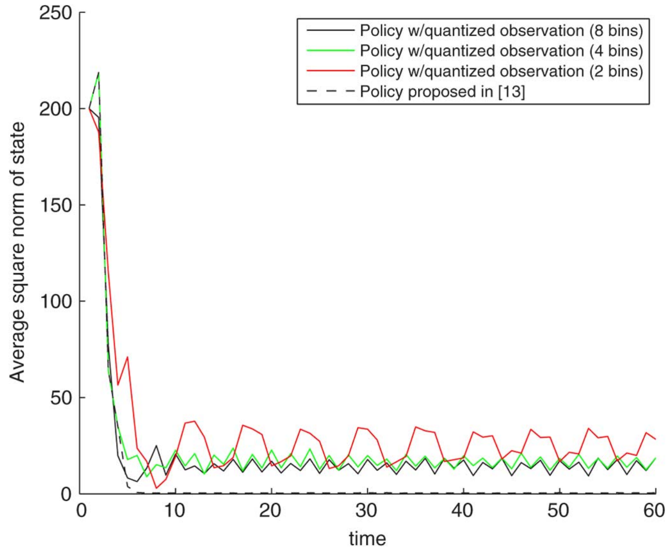

Fig. 2. Empirical average of the square norm of the state under various quantization choices versus the policy in [13].

where $\bar{u}_{\kappa t} := (u_{\kappa t}^\top, \cdots, u_{\kappa(t+1)-1}^\top)^\top \in \mathbb{R}^{\kappa m}$, and $\bar{w}_{\kappa t} := \mathcal{R}_\kappa(A, I)(w_{\kappa t}^\top, \cdots, w_{\kappa(t+1)-1}^\top)^\top$ is zero mean noise. It follows that:

$$\begin{aligned}
&\mathsf{E}^{\mathcal{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \bar{w}_{\kappa t}\| - \|x_{\kappa t}\|\right]\\
&\quad \leqslant \mathsf{E}^{\mathcal{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t}\| - \|x_{\kappa t}\|\right]\\
&\quad\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}.
\end{aligned}$$

Selecting the controls $\bar{u}_{\kappa t} = -\mathcal{R}_\kappa(A,B)^\dagger A^\kappa \mathbf{q}(x_{\kappa t})$ as in (2) and using the fact that $\mathbf{q}(x_{\kappa t}) = \Pi_{x_{\kappa t}}(\mathbf{q}(x_{\kappa t})) + \Pi_{x_{\kappa t}}^\perp(\mathbf{q}(x_{\kappa t}))$, we arrive at

$$\begin{aligned}
&\mathsf{E}^{\mathcal{F}_{\kappa t}}\left[\left\|x_{\kappa(t+1)}\right\| - \|x_{\kappa t}\|\right]\\
&\leqslant \|A^\kappa x_{\kappa t} - A^\kappa \mathbf{q}(x_{\kappa t})\| - \|x_{\kappa t}\|\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}\\
&= \|x_{\kappa t} - \mathbf{q}(x_{\kappa t})\| - \|x_{\kappa t}\|\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4} \quad \text{since } A \text{ is orthogonal}\\
&\leqslant \|x_{\kappa t} - \mathrm{sat}_r(x_{\kappa t})\| - \|x_{\kappa t}\| + \|\mathrm{sat}_r(x_{\kappa t}) - \mathbf{q}(x_{\kappa t})\|\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}\\
&= \|x_{\kappa t} - \mathrm{sat}_r(x_{\kappa t})\| - \|x_{\kappa t}\|\\
&\quad + \left\|\mathrm{sat}_r(x_{\kappa t}) - \Pi_{x_{\kappa t}}\left(\mathbf{q}(x_{\kappa t})\right) - \Pi_{x_{\kappa t}}^\perp\left(\mathbf{q}(x_{\kappa t})\right)\right\|\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}\\
&\leqslant -r + \|\mathrm{sat}_r(x_{\kappa t}) - \Pi_{x_{\kappa t}}\left(\mathbf{q}(x_{\kappa t})\right)\|\\
&\quad + \left\|\Pi_{x_{\kappa t}}^\perp\left(\mathbf{q}(x_{\kappa t})\right)\right\|\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}\\
&\leqslant -r + r\left(1 - \cos(\varphi)\right) + r\sin(\varphi)\\
&\quad + \sqrt{\kappa}\sigma_{\max}\left(\mathcal{R}_\kappa(A,I)\right)\sqrt[4]{C_4}\\
&\leqslant -b \quad \text{for some } b > 0 \text{ by hypothesis a)}.
\end{aligned}$$

The vector $\bar{u}_{\kappa t}$ in (2) satisfies

$$\begin{aligned}
\|\bar{u}_{\kappa t}\| &\leqslant \left\|\mathcal{R}_\kappa(A,B)^\dagger\right\| \|A^\kappa\| \|\mathbf{q}(x_{\kappa t})\|\\
&\leqslant \frac{r}{\sigma_{\min}}\left(\mathcal{R}_\kappa(A,B)\right)\\
&\leqslant U_{\max}.
\end{aligned}$$

Since $\mathsf{E}[\|w_t\|^4] \leqslant C_4$ for each t and since A is orthogonal, we see that for $t \in \mathbb{N}_0$

$$\begin{aligned}
&\mathsf{E}\left[\left|\left\|x_{\kappa(t+1)}\right\| - \|x_{\kappa t}\|\right|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right]\\
&\quad = \mathsf{E}\left[\left|\left\|x_{\kappa(t+1)}\right\| - \|A^\kappa x_{\kappa t}\|\right|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right]\\
&\quad = \mathsf{E}\Big[\left|\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \mathcal{R}_\kappa(A,I)\bar{w}_{\kappa t}\|\right.\\
&\quad\quad\quad \left. - \|A^\kappa x_{\kappa t}\|\right|^4 \mid \{\|x_{\kappa s}\|\}_{s=0}^t\Big]\\
&\quad \leqslant \mathsf{E}\left[\|\mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \mathcal{R}_\kappa(A,I)\bar{w}_{\kappa t}\|^4 \,\middle|\, \{\|x_{\kappa s}\}_{s=0}^t\right]\\
&\quad \leqslant M
\end{aligned}$$

for some $M > 0$. It remains to define $\xi_t := \|x_{\kappa t}\|$ and appeal to Proposition 4 with the above definition of $(\xi_t)_{t\in\mathbb{N}_0}$ to conclude that there exists some $\gamma > 0$, depending on r, x_0, b, and M, such that $\sup_{t\in\mathbb{N}_0}\mathsf{E}[\xi_t^2] = \sup_{t\in\mathbb{N}_0}\mathsf{E}_{x_0}[\|x_{\kappa t}\|^2] \leqslant \gamma$. A standard argument, e.g., as in [13, Proof of Lemma 9], shows that this guarantees $\sup_{t\in\mathbb{N}_0}\mathsf{E}_{x_0}[\|x_t\|^2] \leqslant \gamma'$ for some $\gamma' > 0$. □

III. Numerical Simulation

As a simple example, we consider the system $x_{t+1} = \begin{pmatrix}\cos(\pi/3) - \sin(\pi/3)\\ \sin(\pi/3)\cos(\pi/3)\end{pmatrix} x_t + \begin{pmatrix}1\\0\end{pmatrix} u_t + w_t$, where $x_0 = (10\ \ 10)^\top$ and $w_t \in \mathcal{N}(0, I)$. Here, $\kappa = 2$, $\sigma_{\max}(\mathcal{R}_\kappa(A,I)) = \sqrt{2}$, $C_4 = \mathsf{E}[\|w_t\|^4] = \mathsf{E}[(\chi^2(2))^2] = 8$, and $\sigma_{\min}(\mathcal{R}_\kappa(A,B)) = (\sqrt{2}/2)$. Hence the assumptions of Theorem 2 stipulate that $r > (\sqrt{2}\sqrt{2}\sqrt[4]{8}/\cos(\varphi) - \sin(\varphi))$ and $r \leqslant U_{\max}(\sqrt{2}/2)$. We choose (arbitrarily) the number of bins to be 8, and the quantized point to be located on the bisecting axis of each bin (exactly as in Fig. 1.); then $\varphi = (\pi/8)$, and we obtain $r > (2\sqrt{2}/0.54) = 6.22$ and $U_{\max} \geqslant \sqrt{2}r > 8.8$. Fig. 2. shows the average of the square norm of the state over 1000 runs of the above system under the quantized policy. The result is compared with a similar policy having the same control authority $U_{\max}$ but a lower number of bins than that required by our main Theorem; observe that the controller appears to ensure mean-square boundedness in this particular system. The results are also compared with the policy proposed in [13], which was based on perfect (as opposed to quantized) state information.

References

[1] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, pp. 1279–1289, 2000.

[2] D. Chatterjee, F. Ramponi, P. Hokayem, and J. Lygeros, "On mean square boundedness of stochastic linear systems with bounded controls," *Syst. Control Lett.*, vol. 61, pp. 375–380, 2012.

[3] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Trans. Autom. Control*, vol. 35, pp. 916–924, 1990.

[4] N. Elia and S. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control,*, vol. 46, pp. 1384–1400, 2002.

[5] J. P. Hespanha, P. Naghshtabrizi, and X. Yonggang, "A survey of recent results in networked control systems," in *Proc. IEEE*, 2007, vol. 95, pp. 138–162.

[6] P. Hokayem, D. Chatterjee, F. Ramponi, and J. Lygeros, Stable Networked Control Systems With Bounded Control Authority 2012 [Online]. Available: http://dx.doi.org/10.1109/TAC.2012.2195934

[7] P. Hokayem, E. Cinquemani, D. Chatterjee, F. Ramponi, and J. Lygeros, "Stochastic receding horizon control with output feedback and bounded controls," *Automatica*, vol. 48, pp. 77–88, 2012.

[8] A. S. Matveev and A. V. Savkin, "Shannon zero error capacity in the problems of state estimation and stabilization via noisy communication channels," *Int. J. Control*, vol. 80, pp. 241–255, 2007.

[9] G. Nair and R. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM J. Control Optim.*, vol. 43, pp. 413–436, 2004.
[10] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," in *Proc. IEEE*, 2007, vol. 95, pp. 108–137.
[11] D. Nešić and D. Liberzon, "A unified framework for design and analysis of networked and quantized control systems," *IEEE Trans. Autom. Control*, vol. 54, pp. 732–747, 2009.
[12] R. Pemantle and J. S. Rosenthal, "Moment conditions for a sequence with negative drift to be uniformly bounded in L^r," *Stoch. Processes Appl.*, vol. 82, pp. 143–155, 1999.
[13] F. Ramponi, D. Chatterjee, A. Milias-Argeitis, P. Hokayem, and J. Lygeros, "Attaining mean square boundedness of a marginally stable stochastic linear system with a bounded control input," *IEEE Trans. Autom. Control*, vol. 55, pp. 2414–2418, 2010.
[14] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Trans. Autom. Control*, vol. 49, pp. 1549–1561, 2004.
[15] Y. Tipsuwan and M.-Y. Chow, "Control methodologies in networked control systems," *Control Eng. Practice*, vol. 11, pp. 1099–1111, 2003.
[16] T. C. Yang, "Networked control system: A brief survey," in *Proc. Inst. Elect. Eng.*, 2006, vol. 153, pp. 403–412.
[17] S. Yüksel, "Stochastic stabilization of noisy linear systems with fixed-rate limited feedback," *IEEE Trans. Autom. Control*, vol. 55, pp. 2847–2853, 2010.

On Stability of Systems With Aperiodic Sampling Devices

Chung-Yao Kao and Hisaya Fujioka

***Abstract*—This technical note is concerned with stability analysis of aperiodic sampled-data systems. The stability problem is tackled from a pure discrete-time point of view, where the at-sampling behavior of the system is modelled as the response of a nominal discrete-time LTI system in feedback interconnection with a structured uncertainty. Conditions under which the uncertainty is positive real (PR) are identified. Based on the PR property, a number of integral quadratic constraints (IQC) are derived and the IQC theory is applied to derive stability conditions. Numerical examples are given to illustrate the effectiveness of the proposed approach.**

***Index Terms*—Nonuniform sampling, sampled-data systems, stability.**

I. Introduction

Consider the following state feedback sampled-data system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ u(t) &= Kx(t_k),\ \ \forall t \in [t_k, t_{k+1})\end{aligned} \tag{1}$$

where x and u respectively denote the state and the control input taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, and the sampling sequence $\{t_k\}_{k=0}^{\infty}$ satisfies $0 = t_0 < t_1 < \cdots < t_k < \cdots$, with $\lim_{k\to\infty} t_k = \infty$. When the sampling sequence is uniform, it is well-known that system (1) can be analyzed and/or designed in discrete-time. Such systems have been extensively studied over the past decades and a well-developed and understood theory has been established; see for example [1]. The situation is more complicated when the sampling pattern is nonuniform (and usually unknown *a priori*). In this case, the discretized system has complicated dependency on the uncertain sampling pattern and offers little useful information about stability or performance of the original system. Sampled-data systems with a nonuniform sampling pattern may arise in networked and/or embedded control systems, where resources for measurement and control are limited (see [2], [3] and references therein). In view of the widespread use of networked and embedded control systems, it is both theoretically and practically important to develop tools for accurate robust stability analysis of system model (1) against variation of sampling intervals.

In the literature, several different approaches have been proposed for robust stability analysis of system (1) with a nonuniform sampling pattern. The *impulsive system approach* views system (1) as a hybrid system—a system with both continuous-time and discrete-time parts [4]. Analysis of such systems, particularly in the sampled-data context, can be dated back to the early 1990s (see [5], [6]). For the recent development on the nonuniform sampling, we refer to [7], where stability conditions are derived by Lyapunov stability theory using Lyapunov functions with jumps that correspond to the nonuniform sampling pattern. The *input delay approach* takes a pure continuous-time point of view and models the sampled-data input as the result of a continuous-time signal subject to a sawtooth delay. This approach was first proposed in [8]. Following this approach, stability conditions were derived either by the Lyapunov stability theory [8]–[12] or by operator-theoretic methods such as the scaled small gain theorem [13], [14]. On the other hand, the *discrete-time approach* tackles the stability problem from a pure discrete-time point of view. The behavior of system (1) at sampling instances is examined, which leads to a robust stability problem of a discrete-time system where the variation of the sampling period of system (1) is modelled as a structured uncertainty. Following this approach, stability conditions were derived by the use of robust linear matrix inequalities [15], [16], the scaled small gain theorem [17], [18], and the convex polytopic embedding [19], [20]. Recently, inspired by the advantages of the input delay and the discrete-time approaches, a novel approach is proposed in [21], [22], which utilizes the discrete-time Lyapunov theorem to analyze the continuous-time system (1). With a new type of Lyapunov functional, the new method relaxes certain conditions typically required in the input delay approach.

In this technical note, we follow the discrete-time approach and specifically aim at refining the stability condition proposed in [17] and [18]. The key idea is to identify certain property of the structured uncertainty in the transformed discrete-time system that is useful for robust stability analysis. More specifically, we discover that, under certain conditions, the structured uncertainty exhibits the so-called "positive real" (PR) property. Based on the PR property, a number of integral quadratic constraints (IQC) are derived, which eventually lead to a tighter robust stability condition for system (1).

Notation and Terminology

Symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^{n\times m}$, $\mathbb{C}$, $\mathbb{C}_+$, $\mathbb{C}^{n\times m}$, and $\mathbb{Z}_+$ are used to denote respectively the sets of real numbers, nonnegative real numbers, $n\times m$ real matrices, complex numbers, complex numbers with nonnegative real part, $n \times m$ complex matrices, and nonnegative integers. Given a matrix M, the transposition and the conjugate transposition of M are denoted by M' and M^*, respectively. The spectrum of M is denoted as $\mathrm{eig}(M)$ while the spectral radius of M is denoted as $\rho(M)$. Symbol l_2 denotes the space of $\mathbb{R}^n$-valued, square summable functions

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C.-Y. Kao is with the Department of Electrical Engineering, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan, R.O.C. (e-mail: cykao@mail.nsysu.edu.tw).
H. Fujioka, deceased, was with the Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan.

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