Fast-paced review of continuous-time systems

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Rev. 1.1, 2019-10-07

0.1 Autonomous nonlinear systems

A time-invariant *autonomous* system (I'll make liberal use of the word *system* to mean the *model* of a physical system) in state-space form is a [system of] first order differential equation[s]:

$$\dot{x}(t) = \bar{f}(x(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ (state vector) and $\bar{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$. Here "autonomous" means that the system has no input. In the sequel, t_0 will denote an "initial time", and $x(t_0) = x_0$ will be an "initial condition" (or "initial state"); as is customary in system theory, later we will let $t_0 = 0$. The differential equation together with the initial condition forms a so-called Cauchy problem; without further mention, we assume that conditions for the existence and uniqueness of a solution to (1), at least in a neighborhood of t_0 but wishfully for all $t \geq t_0$, are satisfied. A point $\bar{x}$ is called an *equilibrium point* of (1) if $\bar{f}(\bar{x}) = 0$; if $x_0 = x(t_0) = \bar{x}$, then the unique solution is the constant function $x(t) = \bar{x}$ for all t (the system remains in the same state forever).

The equilibrium point $\bar{x}$ is *stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\|x(t_0) - \bar{x}\| \leq \delta$, $\|x(t) - \bar{x}\| \leq \epsilon$ for all $t \geq 0$. In words, if the initial state is close enough to the equilibrium point, the state remains close to the equilibrium point forever.
Example: pendulum without friction $\begin{bmatrix} \dot{\vartheta}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} \omega(t) \\ -g/l \cdot \sin(\vartheta(t)) \end{bmatrix}$.
It can be shown that $\bar{x} = \begin{bmatrix} \bar{\vartheta} \\ \bar{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable equilibrium point.
Another example: *LC* oscillator with no resistance (and hence no energy dissipation); take as state variables the charge in the capacitor and the current throughout the inductor (the equilibrium point is the origin: no charge, no current. This system happens to be *linear*).

The equilibrium point $\bar{x}$ is *asymptotically stable* if it is stable and, moreover, there exists a $\delta > 0$ such that, for all initial states such that $\|x(t_0) - \bar{x}\| \leq \delta$, it holds $\lim_{t\to+\infty} x(t) = \bar{x}$. In words, if the initial state is close enough to the equilibrium point, the state remains close and eventually converges to it. Examples: pendulum *with* friction, *RLC* oscillator; same equilibrium points as before. If an equilibrium point is not stable it is called (guess what?) *unstable*.

0.2 Nonlinear systems with input

A single-input, single-output time-invariant system in state-space form is a system of equations:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = h(x(t)), \end{cases} \tag{2}$$

where $u(t) \in \mathbb{R}$ is an input signal, $y(t) \in \mathbb{R}$ is an output signal, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the evolution map, and $h : \mathbb{R}^n \to \mathbb{R}$ is a state-to-output map. With respect to a reference constant input $u(t) \equiv \bar{u}$, a state $\bar{x}$ is called an equilibrium point if $f(\bar{x}, \bar{u}) = 0$. To justify this definition, let $\bar{f}(x(t)) := f(x(t), \bar{u})$ and compare with the definition for (1). With the same setting, $\bar{x}$ is called respectively *stable*, *asymptotically stable*, or *unstable* (with respect to the nominal input $\bar{u}$) depending on what happens to the solution of the autonomous system $\dot{x}(t) = \bar{f}(x(t))$.

0.3 Linear systems

A time-invariant autonomous *linear* system in state-space form is a first order linear differential equation with constant matrix:

$$\dot{x}(t) = Ax(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.

The definitions of equilibrium point (stable, asymptotically stable, unstable) are exactly as in the general case (Section 0.1). Some facts follow:

- The equilibrium points of a linear system form a subspace of $\mathbb{R}^n$. Indeed if $\bar{x}$ is an equilibrium point ($A\bar{x} = 0$) and $\alpha \in \mathbb{R}$, then also $A(\alpha\bar{x}) = \alpha(A\bar{x}) = 0$, so that $\alpha\bar{x}$ is an equilibrium point; and if $\bar{x}_1, \bar{x}_2$ are equilibrium points, then $A(\bar{x}_1+\bar{x}_2) = A\bar{x}_1+A\bar{x}_2 = 0$, so that $\bar{x}_1+\bar{x}_2$ is an equilibrium point. Trivial example: if $A = 0$, every $\bar{x} \in \mathbb{R}^n$ is an equilibrium point; more generally, the equilibrium subspace is the subspace null $A = \{x \in \mathbb{R}^n \ : \ Ax = 0\}$.

- If an equilibrium point is stable, then any other equilibrium point is stable.
  Indeed, suppose that $\bar{x}_1$ is a stable equilibrium point. This means that $A\bar{x}_1 = 0$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\|x(0) - \bar{x}_1\| \leq \delta$, then $\|x(t) - \bar{x}_1\| \leq \varepsilon$ for all $t \geq 0$. Now let $\bar{x}_2$ be another equilibrium point ($A\bar{x}_2 = 0$), and define $x_2(t) = x(t) - \bar{x}_1 + \bar{x}_2$; by linearity of the system, $x_2(t)$ is the solution of (3) corresponding to the initial condition $x(0) - \bar{x}_1 + \bar{x}_2$: indeed $x_2(0) = x(0) - \bar{x}_1 + \bar{x}_2$ by definition, and

  $$\dot{x}_2(t) = \dot{x}(t) = Ax(t) = Ax(t) - A\bar{x}_1 + A\bar{x}_2 = A(x(t) - \bar{x}_1 + \bar{x}_2) = Ax_2(t).$$

  Then the stability statement above reads: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\|x(0)-\bar{x}_1\| = \|x_2(0) - \bar{x}_2\| \leq \delta$, then $\|x(t) - \bar{x}_1\| = \|x_2(t) - \bar{x}_2\| \leq \varepsilon$ for all $t \geq 0$: hence $\bar{x}_2$ is also stable.

- Only the origin can be an *asymptotically* stable equilibrium point, and in this case the origin is also *the only equilibrium point of the system.*
  Indeed, let 0 be asymptotically stable: then for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\|x(0)\| \leq \delta$, then $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$, and moreover $\lim_{t\to+\infty} x(t) = 0$. If there existed another equilibrium point $\bar{x}$, then $\lambda\bar{x}$ would also be an equilibrium point ($A(\lambda\bar{x}) = \lambda A\bar{x} = 0$) for all $\lambda \in \mathbb{R}$; but for small enough λ, it would hold $\|\lambda\bar{x}\| \leq \delta$: hence pretending that $\bar{x}$ is an equilibrium point (if $x(0) = \bar{x}$, then $x(t) = \bar{x}$ for all t) would contradict the convergence to the origin.
  When the origin is an asymptotically stable equilibrium point, it is customary to say that *the system* is asymptotically stable.

A single-input, single-output time-invariant linear system in state-space form is a pair:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{4}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $B \in \mathbb{R}^{n\times 1}$ (a column), and $C \in \mathbb{R}^{1\times n}$ (a row).

0.4 Where do linear systems come from?

Linear system may be models of physical systems that are indeed linear (for instance passive electrical networks; an input is, for example, a voltage source connected to some node of the network, and an output signal can be the voltage at some other node). But in many cases they come up as the linearization of a non-linear system like (2) around an equilibrium point $\bar{x}$ and of a nominal constant input $\bar{u}$.

Indeed, let $u(t) = \bar{u} + \Delta u(t)$, $x(t) = \bar{x} + \Delta x(t)$, and $y(t) = h(\bar{x}) + \Delta y(t)$. Approximating (2) with a first-degree Taylor polynomial,

$$\begin{aligned}
\dot{x}(t) &\simeq f(\bar{x}, \bar{u}) + \left[\frac{\partial f}{\partial x}\right]_{\substack{x=\bar{x}\\u=\bar{u}}} \Delta x(t) + \left[\frac{\partial f}{\partial u}\right]_{\substack{x=\bar{x}\\u=\bar{u}}} \Delta u(t) \\
\frac{d}{dt}\Delta x(t) &\simeq 0 + \left[\frac{\partial f}{\partial x}\right]_{\substack{x=\bar{x}\\u=\bar{u}}} \Delta x(t) + \left[\frac{\partial f}{\partial u}\right]_{\substack{x=\bar{x}\\u=\bar{u}}} \Delta u(t) := A\ \Delta x(t) + B\ \Delta u(t) \\
y(t) &\simeq h(\bar{x}) + \left[\frac{\partial h}{\partial x}\right]_{x=\bar{x}} \Delta x(t) \\
\Delta y(t) &\simeq \left[\frac{\partial h}{\partial x}\right]_{x=\bar{x}} \Delta x(t) := C\ \Delta x(t)
\end{aligned}$$

Re-labeling the signals (e.g. x instead of Δx) the system linearized around $\bar{x}, \bar{u}$ reads exactly as (4). Here is an *amazing* fact about the linearized system:

- if the system linearized around $\bar{x}$ turns out to be asymptotically stable, then the equilibrium point $\bar{x}$ is asymptotically stable *for the original nonlinear system.*

(But nothing can be said, in general, if the linearized system is just stable.) This fact follows from the so-called Lyapunov stability theory, and is a very powerful tool, because as we shall see checking the asymptotic stability of the linearized system is far easier than it is to check the stability of $\bar{x}$ with respect to the nonlinear one.

0.5 The exponential matrix

Recall the definition of the exponential function ($z \in \mathbb{C}$):

$$e^z = \exp(z) := \sum_{k=0}^{+\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$

Here z is a complex number. But a very similar definition of "exponential" can be applied to *operators*, and in our case to *square matrices*, that can be interpreted as "generalizations of numbers". Let $A \in \mathbb{R}^{n \times n}$. We define:

$$e^A = \exp(A) := \sum_{k=0}^{+\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots,$$

where I is the identity matrix (the generalization of the number 1). This definition is well posed for all $A \in \mathbb{R}^{n \times n}$. Some facts follow:

- If 0 is the zero matrix, $e^0 = I$.
- If $A, B \in \mathbb{R}^{n \times n}$ and A, B commute ($AB = BA$), then $e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$.
- For all $A \in \mathbb{R}^{n \times n}$, e^A is invertible and its inverse is e^{-A}. (It follows immediately from the previous points.)
- If $A \in \mathbb{R}^{n \times n}$ and $t \in \mathbb{R}$, $\frac{d}{dt} e^{At} = A \cdot e^{At}$.

Example. Suppose that A is diagonalizable: there exists an invertible matrix T (change of basis) such that $T^{-1}AT = D$ (hence $A = TDT^{-1}$), where D is a diagonal matrix:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}.$$

Then,

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \ldots = TT^{-1} + TDT^{-1}t + \frac{TD^2T^{-1}t^2}{2} + \frac{TD^3T^{-1}t^3}{3!} + \ldots \\
&= T\left(I + Dt + \frac{D^2t^2}{2} + \frac{D^3t^3}{3!} + \ldots\right)T^{-1} \\
&= T\left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & \\ & \lambda_2 t & \\ & & \lambda_3 t \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2t^2}{2} & & \\ & \frac{\lambda_2^2t^2}{2} & \\ & & \frac{\lambda_3^2t^2}{2} \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^3t^3}{3!} & & \\ & \frac{\lambda_2^3t^3}{3!} & \\ & & \frac{\lambda_3^3t^3}{3!} \end{bmatrix} + \ldots\right)T^{-1} \\
&= T\begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{bmatrix}T^{-1} \qquad \left(= T\ e^{Dt}\ T^{-1}\right).
\end{aligned}
$$

Facts:

- If we know how to diagonalize A, we know how to compute the exponential e^{At}.
- λ_1, λ_2, and λ_3 are the eigenvalues of A (and the columns of T form a linearly independent set of corresponding eigenvectors). Thus, the "intrinsic" behaviour of e^{At} is essentially determined by the eigenvalues of A.

The general case (matrices that cannot be diagonalized) requires the so-called canonical Jordan form, but is otherwise quite straightforward, because a Jordan form can be thought of as the sum of two commuting matrices, of which one is diagonal and the other is such that its exponential amounts to a finite sum.

0.6 Solution of the linear system

The general solution of the autonomous system (3) is

$$x(t) = e^{A(t-t_0)}\ x_0. \tag{5}$$

Indeed the above $x(t)$ satisfies (3), including the initial condition:

$$
\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}e^{A(t-t_0)}x_0 = Ae^{A(t-t_0)}x_0 = Ax(t); \\
x(t_0) &= e^{A(t_0-t_0)}x_0 = e^0x_0 = x_0.
\end{aligned}
$$

Now, checking the asymptotic stability of the linear system (3) is just a matter of checking the behaviour of (5). Introduce a change of basis in the state space, letting $x(t) = T\xi(t)$, where T is the matrix that diagonalizes A:

$$
\begin{aligned}
x(t) &= e^{A(t-t_0)}\ x_0; \\
T\xi(t) &= e^{A(t-t_0)}\ T\xi_0; \\
\xi(t) &= T^{-1}\ e^{A(t-t_0)}\ T\xi_0 = T^{-1}\ Te^{D(t-t_0)}\ T^{-1}T\xi_0 = \begin{bmatrix} e^{\lambda_1(t-t_0)} & & \\ & e^{\lambda_2(t-t_0)} & \\ & & e^{\lambda_3(t-t_0)} \end{bmatrix}\xi_0.
\end{aligned}
$$

The functions $e^{\lambda_i(t-t_0)}$ appearing in the above expression are called "modes" of the system. Since A is real, its eigenvalues λ_i are either real, or they come in conjugate pairs $\lambda_{12} = a \pm j\omega$. If an eigenvalue is real and negative, say $\lambda_3 = -\alpha < 0$, the corresponding mode is an exponential decay, $e^{-\alpha(t-t_0)}$; if two conjugate eigenvalues have negative real part, say $\lambda_{12} = -\alpha \pm j\omega$ the corresponding modes are complex-valued functions, but two suitable combinations of them yield *damped* oscillatory modes: $e^{-\alpha(t-t_0)}\sin(\omega(t-t_0))$, $e^{-\alpha(t-t_0)}\cos(\omega(t-t_0))$.

Thus, if all the eigenvalues λ_i have negative real part, then $\xi(t) \to 0$ as $t \to \infty$, and so does $x(t) = T\xi(t)$, irrespective of the initial state x_0. Indeed, (3) is asymptotically stable if and only if A has all its eigenvalues in the open half-plane $\{\Re z < 0\}$. Vice versa, if at least one of the eigenvalues has positive real part then at least one component of $\xi(t)$ diverges exponentially as $t \to \infty$, so does at least one component of $x(t)$, and the system is unstable. The case where one or more eigenvalues lie on the imaginary axis is somewhat subtler, and we shall not deal with it.

The general solution of the differential equation in system (4) is

$$x(t) = e^{A(t-t_0)}\, x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau. \tag{6}$$

Indeed the above $x(t)$ satisfies (4), including the initial condition:

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt} e^{A(t-t_0)} x_0 + \frac{d}{dt}\int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau \\
&= A e^{A(t-t_0)} x_0 + \int_{t_0}^{t} \frac{d}{dt} e^{A(t-\tau)} B u(\tau) d\tau + e^{A(t-t)} B u(t) \\
&= A e^{A(t-t_0)} x_0 + \int_{t_0}^{t} A e^{A(t-\tau)} B u(\tau) d\tau + e^{A(t-t)} B u(t) \\
&= A\left(e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau \right) + B u(t) = A x(t) + B u(t); \\
x(t_0) &= e^{A(t_0-t_0)} x_0 + \int_{t_0}^{t_0} e^{A(t-\tau)} B u(\tau) d\tau = e^0 x_0 + 0 = x_0.
\end{aligned}$$

Once the evolution state is known, that of the output is trivial:

$$y(t) = Cx(t) = Ce^{A(t-t_0)}\, x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)} B u(\tau)\, d\tau \tag{7}$$

You can see from (7) that $y(t)$ is the sum of two contributions: the first one $Ce^{A(t-t_0)}\, x_0$ is due *only* to the initial state, and is called *free evolution*; the second one $\int_{t_0}^{t} Ce^{A(t-\tau)} B u(\tau)\, d\tau$ is due *only* to the input from t_0 to t, and is called *forced response*. The function

$$h(t) := Ce^{At} B \tag{8}$$

is the so-called *impulse response* of the system, because it is the forced response corresponding to a “Dirac's delta” $\delta(t)$ fed as the input; indeed, letting $x_0 = 0$, (7) reads

$$y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)} B \delta(\tau)\, d\tau = Ce^{At} B = h(t).$$

Summing up, the forced response is a convolution of the input with the impulse response: $\int_{t_0}^{t} h(t-\tau) u(\tau)\, d\tau$.

0.7 Meanwhile, in the Laplace domain...

Even neglecting any sort of mathematical rigor, we should recall at least three fundamental facts about the “unilateral” Laplace transform of a function $x(t)$:

- Its definition: $X(s) = \mathcal{L}[x](s) := \int_0^\infty x(t)e^{-st}\ dt$.
- The rule for the transform of a derivative (of a well-behaved function such that $x(t)e^{-st} \to 0$ as $t \to \infty$ at least for one $s \in \mathbb{C}$): integrating by parts,

$$\begin{aligned}\int_0^\infty \dot{x}(t)e^{-st}\ dt &= \left[x(t)e^{-st}\right]_0^\infty - \int_0^\infty x(t)(-s)e^{-st}\ dt = -x(0) + s\int_0^\infty x(t)e^{-st}\ dt \\ &= -x(0) + sX(s)\end{aligned}$$

  Take-home message: *except for the "initial condition"* $x(0)$, the Laplace variable s plays in the Laplace domain the role that in the time domain pertains to the operator $\frac{d}{dt}$.

- The rule for the transform of a *convolution*: if $y(t) = \int_0^t h(t-\tau)u(\tau)\ d\tau$, then

$$Y(s) = H(s) \cdot U(s).$$

  Take-home message: convolutions in the time domain become products in the Laplace domain.

Now consider again our linear system (4),

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t),\end{cases}$$

and for the sake of simplicity let $t_0 = 0$. The corresponding solution (7) reads

$$y(t) = Ce^{At}\ x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\ d\tau \tag{9}$$

Let's do some *sporting club* mathematics. Take, side by side, the transform of the differential equation in (4):

$$\begin{aligned}\mathcal{L}[\dot{x}](s) &= \mathcal{L}[Ax](s) + \mathcal{L}[Bu](s); \\ -x(0) + sX(s) &= AX(s) + BU(s); \\ (sI - A)\, X(s) &= x_0 + BU(s); \\ X(s) &= (sI - A)^{-1}\, x_0 + (sI - A)^{-1}\, BU(s).\end{aligned}$$

(We're playing with vector- and matrix-valued functions here, not with scalar functions—don't worry, it works anyway.) Now turn to the transform of the output equation:

$$\begin{aligned}Y(s) &= \mathcal{L}[y](s) = \mathcal{L}[Cx](s) \\ &= C\,(sI - A)^{-1}\, x_0 + C\,(sI - A)^{-1}\, B \cdot U(s).\end{aligned} \tag{10}$$

- Fact: the matrix-valued complex function $(sI - A)^{-1}$ is the Laplace transform of the matrix exponential e^{At}, in pretty much the same way as the scalar function $\frac{1}{s-a}$ is the transform of e^{at}.
- Fact: the *scalar* function $H(s) = C\,(sI - A)^{-1}\, B$ is the Laplace transform of the impulse response $h(t) = Ce^{At}B$. It is called the *transfer function* of the linear system.

It remains to notice that the rightmost term in (10) is a product $H(s) \cdot U(s)$, so it is the transform of a convolution. Now enjoy recognizing in (10) the Laplace transform of (9).

0.8 What about the transfer function, then?

Remember: $H(s) = C\,(sI - A)^{-1}\,B$ is a *scalar* function, since C is a row and B is a column. Understanding its structure boils down to recalling what it is to compute the inverse of a matrix. (A matrix of numbers? a matrix of functions? it doesn't matter.)

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where c_{ij} are the so-called *co-factors*: $\pm$ determinants of minors of order $2 = n - 1$. Then

$$(sI - A)^{-1} = \begin{bmatrix} s - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & s - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & s - a_{33} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} c_{11}(s) & c_{12}(s) & c_{13}(s) \\ c_{21}(s) & c_{22}(s) & c_{23}(s) \\ c_{31}(s) & c_{32}(s) & c_{33}(s) \end{bmatrix}}{\det\,(sI - A)} = \frac{N(s)}{\det\,(sI - A)}$$

where now the cofactors $c_{ij}(s)$ in the matrix $N(s)$ are functions of s: actually, since they are determinants of minors of order $n - 1$, all of them are *polynomials* of degree at most $n - 1$. Finally,

$$H(s) = C(sI - A)^{-1}B = \frac{CN(s)B}{\det\,(sI - A)} = \frac{b(s)}{a(s)}.$$

The numerator $b(s) = CN(s)B$ is a *polynomial* of degree at most $n - 1$. The denominator $a(s) = \det\,(sI - A)$ is a *monic* polynomial of order n that you should recognize from basic linear algebra: it is called the *characteristic polynomial* of A, and its roots are the eigenvalues of A. Summing up, $H(s)$ is a strictly proper rational transfer function:

$$\begin{aligned} H(s) &= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \\ &= b_{n-1}\frac{(s - z_1)(s - z_2)\ldots(s - z_{n-1})}{(s - p_1)(s - p_2)\ldots(s - p_n)} \end{aligned}$$

The "poles" $p_1, p_2, \ldots, p_n$ coincide, one by one and counting their multiplicity, with the eigenvalues of A, unless there is some zero-pole cancellation. In the latter case, we can only say that each pole is also an eigenvalue of A, but the set of poles (counting multiplicity) does not exhaust the set of eigenvalues (counting multiplicity). Zero-pole cancellations happen when the system is "not completely reachable" or "not completely observable", and indeed the poles coincide with the eigenvalues of the "reachable and observable subsystem"; but this is a subject for another day.

0.9 From continuous time to discrete time

Suppose that for $k \in \mathbb{Z}$ the input $u(t)$ is constant over intervals of the form $[k\Delta, (k + 1)\Delta)$, i.e. $u(t) \equiv u_k$ over $[k\Delta, (k + 1)\Delta)$. Then the link between input, state, and output enforced by (4) is well represented by a *discrete-time* counterpart where all the signals involved are *sequences of numbers or vectors* u_k, x_k, and y_k ($k \in \mathbb{Z}$), representing *sampled* versions of $u(t)$, $x(t)$, and $y(t)$ respectively, with sampling time Δ. All that it takes to work out such a *discretized* system is to exploit the solution (6)-(7) by letting $t_0 = k\Delta$ and taking into account the constraint on the input. Let $x_k := x(k\Delta)$ (in particular $x_0 := x(0)$. It holds:

$$
\begin{aligned}
x_{k+1} = x((k+1)\Delta) &= e^{A((k+1)\Delta - k\Delta)}\ x(k\Delta) + \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\tau)} Bu(\tau)\ d\tau \quad \text{(recall } u(\tau) \equiv u_k\text{)} \\
&= e^{A\Delta}\ x_k + \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\tau)} B\ d\tau \cdot u_k \quad \text{(let } \sigma = (k+1)\Delta - \tau, d\sigma = -d\tau\text{)} \\
&= e^{A\Delta}\ x_k + \int_0^{\Delta} e^{A\sigma} B\ d\sigma \cdot u_k \\
y_k = y(k\Delta) = Cx(k\Delta) = Cx_k
\end{aligned}
\tag{11}
$$

Defining the new matrices $\bar{A} := e^{A\Delta}$, $\bar{B} := \int_0^{\Delta} e^{A\sigma} B\ d\sigma$, and $\bar{C} := C$, we get to the desired discretization:

$$
\begin{cases} x_{k+1} = \bar{A}x_k + \bar{B}u_k, \\ y_k = \bar{C}x_k. \end{cases} \tag{12}
$$

This is a single-input, single-output, time-invariant, discrete-time linear system in state-space form. Despite the pedantic name, it is nothing more than a *recursive algorithm* that updates the state from the current state x_k and an input sample u_k, and maps the current state linearly to an output sample y_k.

Note: I don't like very much the notation "x_k" with the subscript, and anyway I prefer to denote time always with the letter t. So, since during the course we will deal mostly with discrete-time systems, if this does not generate confusion I will make liberal use of the following notation:

$$
\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{13}
$$

which is exactly the same scheme with some cosmetics in notation. If you prefer, you can interpret (13) as a sampled version of a continuous-time system (having different matrices instead of A, B, C, of course) with sampling time $\Delta = 1$.

The solution of the recursion in (13) is the following sequence:

$$
x(t) = A^t x_0 + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau). \tag{14}
$$

(The corresponding expression for $y(t)$ is obvious.) To prove (14), use mathematical induction:

- $x(0) = A^0 x_0 + \sum_{\tau=0}^{-1} A^{t-1-\tau} Bu(\tau) = x_0$, so the base of the induction is OK;

- suppose that the formula is true for t: $x(t) = A^t x_0 + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau)$; then

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
&= A\left(A^t x_0 + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau)\right) + I \cdot Bu(t) \\
&= A^{t+1} x_0 + \sum_{\tau=0}^{t-1} A^{t-\tau} Bu(\tau) + A^{(t-t)} \cdot Bu(t) \\
&= A^{t+1} x_0 + \sum_{\tau=0}^{t} A^{t-\tau} Bu(\tau) \\
&= A^{t+1} x_0 + \sum_{\tau=0}^{(t+1)-1} A^{(t+1)-1-\tau} Bu(\tau),
\end{aligned}
$$

  so the formula is true also for $t+1$, and the inductive step is also OK.