Stability and self-optimality of the self-tuning regulator

F. A. Ramponi
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This document contains an asymptotic analysis of the properties of self-tuning regulator. As far as the final exam is concerned, the final take-home message is requested as part of the program, but the tedious computations in the convergence analysis are not.

References:
1) Sergio Bittanti and Marco C. Campi. *Least squares based self-tuning control systems.* In: Identification, Adaptation, Learning. The science of learning models from data (S. Bittanti and G. Picci eds.). Springer-Verlag NATO ASI series - Computer and systems sciences, pages 339-365, 1996.
2) Private notes by Simone Garatti.

0.1 Real system, frozen systems and imaginary system

You should recall that the self-tuning regulator, regarded as the complex control unit + tuning unit, is a *non-linear* and *time-invariant* system. However, to carry on with the analysis, it is convenient to exclude the tuning unit from the picture; in this way, the control unit can be thought of a linear *time-varying* system.

Assumption 0.1.1

The plant is deterministic, and there exists a "true" model in the model class that describes it perfectly. In other words, there exists a parameter $\bar{\theta}$ that explains the measures exactly:

$$y_t = \varphi_t^\top \bar{\theta} \qquad \text{for all } t.$$

(The final result holds true under fairly general conditions, *mutatis mutandis*, also if a process noise is present, if the regressors are random, and so on.)

To fix ideas, we assume that the plant is an ARX$(3,3)$ model; the generalization to ARX(n,m) is immediate. Hence, here is the "true" system:

$$\begin{aligned} y(t) &= \bar{a}_1 y(t-1) + \bar{a}_2 y(t-2) + \bar{a}_3 y(t-3) + \bar{b}_1 u(t-1) + \bar{b}_2 u(t-2) + \bar{b}_3 u(t-3) \\ &= \varphi_t^\top \bar{\theta}, \end{aligned}$$

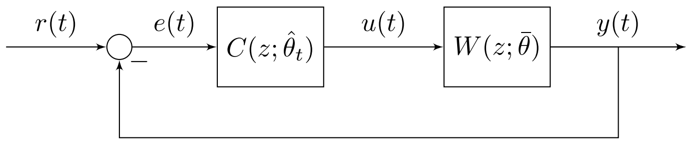
where

$$\begin{aligned} \varphi_t^\top &= \begin{bmatrix} y(t-1) & y(t-2) & y(t-3) & u(t-1) & u(t-2) & u(t-3) \end{bmatrix}, \\ \bar{\theta} &= \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix}^\top. \end{aligned}$$

Therefore, the tuning unit and the control unit are both targeted at the model class ARX$(3,3)$, and in particular the "true parameter" $\bar{\theta}$, the regressors φ_τ fed to the tuning unit, and the estimates $\hat{\theta}_t$ that it produces, all belong to $\mathbb{R}^6$.

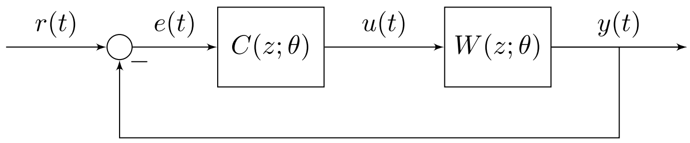
As noted above, for the analysis it is convenient to consider the estimates $\hat{\theta}_t$ as "coming from outside" even if they truly are state variables of the self-tuning regulator, and to think at the closed loop *without* the tuning unit but with a time-varying controller depending on $\hat{\theta}_t$. This is the picture:

Real system $\Sigma(\hat{\theta}_t, \bar{\theta})$



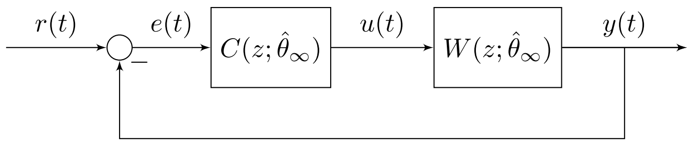
Here, the controller depends on $\hat{\theta}_t$ and the plant depends (by assumption) on the unknown $\bar{\theta}$. At this point, we enlarge the picture by considering different, purely hypothetical systems with the same structure, but where both the controller and the plant depend on the *same* parameter. If we let both the controller and the plant depend on an arbitrary, fixed parameter θ we obtain a so-called *frozen* system:

System frozen at θ: $\Sigma(\theta, \theta)$



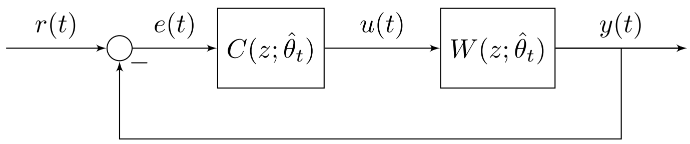
In particular, we may "freeze" the system at the fixed parameter $\theta = \hat{\theta}_\infty$ (we know that this limit exists!):

System frozen at infinity: $\Sigma(\hat{\theta}_\infty, \hat{\theta}_\infty)$



Any "frozen" system is, by definition, linear and time-invariant. But at this point nothing prevents us from plugging in the estimate $\hat{\theta}_t$, *both in the controller and in the plant*; we obtain the so-called "imaginary" system:

Imaginary system: $\Sigma(\hat{\theta}_t, \hat{\theta}_t)$



0.2 The imaginary system behaves like the system frozen at infinity

Assumption 0.2.1

1. The controller design procedure attains internal stability and reference tracking for the system frozen at infinity $\Sigma(\hat{\theta}_\infty, \hat{\theta}_\infty)$. This is fairly general, because the design procedures that we have seen do the job for all θ except for pathological cases (e.g. if the plant "frozen at infinity" has zeros on the unit circumference).

2. The controller design procedure is "continuous", meaning that if $\theta \to \hat{\theta}_\infty$ the corresponding inputs and outputs in the closed loop satisfy

$$\begin{aligned} u_\theta(t) &\to u_\infty(t) \\ y_\theta(t) &\to y_\infty(t) \end{aligned}$$

   for all t. (The subscript ∞ in $y_\infty(t)$ means "corresponding to $\hat{\theta}_\infty$; is is short for $y_{\hat{\theta}_\infty}(t)$.) This is also fairly general and you can give it for granted.

3. The reference signal $\{r(t)\}$ is *bounded*, i.e. there exists $K \in \mathbb{R}$ such that $|r(t)| \leq K$ for all t. This may seem limiting, because we have seen how to design polynomial controllers in order to track correctly *ramps*, *quadratics* and so on; but really, no signal in practical use is really *unbounded*, is it? "Tracking ramps" really means, in practice, tracking decently signals that behave *locally* like ramps, for example triangular waves. (On the other hand, of course, constant references and sinusoids are OK.)

Under the above assumption we show that the imaginary system $\Sigma(\hat{\theta}_t, \hat{\theta}_t)$ behaves, asymptotically, as the system frozen at infinity.

To do this, first let me rewrite the models of the controller and of the plant as functions of z^{-1} instead of z. I mean this:

$$\begin{aligned} W(z;\theta) &= \frac{b(z;\theta)}{a(z;\theta)} = \frac{b_1 z^2 + b_2 z + b_3}{z^3 - a_1 z^2 - a_2 z - a_3} \qquad \text{(divide above and below by } z^3\text{)} \\ &= \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}} := \frac{\beta(z^{-1};\theta)}{\alpha(z^{-1};\theta)} \end{aligned}$$

We do the same for the plant and the controller of the imaginary system:

$$\begin{aligned} W(z;\hat{\theta}_t) &= \frac{b(z;\hat{\theta}_t)}{a(z;\hat{\theta}_t)} = \frac{\beta(z^{-1};\hat{\theta}_t)}{\alpha(z^{-1};\hat{\theta}_t)}; \\ C(z;\hat{\theta}_t) &= \frac{q(z;\hat{\theta}_t)}{p(z;\hat{\theta}_t)} = \frac{\rho(z^{-1};\hat{\theta}_t)}{\pi(z^{-1};\hat{\theta}_t)}. \end{aligned}$$

With this notation, the imaginary system reads:

$$\Sigma(\hat{\theta}_t, \hat{\theta}_t) : \begin{cases} \alpha(z^{-1};\hat{\theta}_t) y(t) = \beta(z^{-1};\hat{\theta}_t) u(t) \\ \pi(z^{-1};\hat{\theta}_t) y(t) = \rho(z^{-1};\hat{\theta}_t) (\underbrace{r(t) - y(t)}_{e(t)}) \end{cases}$$

$$: \begin{cases} y(t) = (1 - \alpha(z^{-1};\hat{\theta}_t)) y(t) + \beta(z^{-1};\hat{\theta}_t) u(t) \\ \quad = a_{1(t)} y(t-1) + a_{2(t)} y(t-2) + a_{3(t)} y(t-3) + b_{1(t)} u(t-1) + b_{2(t)} u(t-2) + b_{3(t)} u(t-3) \\ \quad = \varphi_t^\top \hat{\theta}_t \\ u(t) = (1 - \pi(z^{-1};\hat{\theta}_t)) u(t) - \rho(z^{-1};\hat{\theta}_t) y(t) + \underbrace{\rho(z^{-1};\hat{\theta}_t) r(t)}_{:= r^*(t)} \end{cases}$$

Note: since $r(t)$ is bounded, the (moving average) signal

$$\begin{aligned} r^*(t) &:= \rho(z^{-1};\hat{\theta}_t) r(t) = (b_{1(t)} z^{-1} + b_{2(t)} z^{-2} + b_{3(t)} z^{-3}) r(t) \\ &= b_{1(t)} r(t-1) + b_{2(t)} r(t-2) + b_{3(t)} r(t-3) \end{aligned}$$

is also bounded, because $b_{1(t)}, b_{2(t)}, b_{3(t)}$ are some components of the vector $\hat{\theta}_t$, *that converges* (a convergent sequence is always bounded).

Let's rewrite $\Sigma(\hat{\theta}_t, \hat{\theta}_t)$ in compact form, denoting $y_{\text{im}}(t), u_{\text{im}}(t)$ the signals to avoid confusion later:

$$\Sigma(\hat{\theta}_t, \hat{\theta}_t) : \left\{ \begin{bmatrix} y_{\text{im}}(t) \\ u_{\text{im}}(t) \end{bmatrix} = \begin{bmatrix} 1 - \alpha(z^{-1};\hat{\theta}_t) & \beta(z^{-1};\hat{\theta}_t) \\ -\rho(z^{-1};\hat{\theta}_t) & 1 - \pi(z^{-1};\hat{\theta}_t) \end{bmatrix} \begin{bmatrix} y_{\text{im}}(t) \\ u_{\text{im}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ r^*(t) \end{bmatrix} \right.$$

This is a multi-input multi-output (MIMO) system written in transfer function form; we haven't coped with such systems in the course, so please rely on your intuition. All that matters, here, is that the above one is a *causal*, linear, *time-varying* system that can be realized, in some way, also in state-space form:

$$\Sigma(\hat{\theta}_t, \hat{\theta}_t) : \begin{cases} \xi(t+1) = A_t \xi(t) + B_t \begin{bmatrix} 0 \\ r^*(t) \end{bmatrix} \\ \begin{bmatrix} y_{\text{im}}(t) \\ u_{\text{im}}(t) \end{bmatrix} = C_t \xi(t) + D_t \begin{bmatrix} 0 \\ r^*(t) \end{bmatrix} \end{cases} \tag{1}$$

where $\xi(t)$ is a state vector with suitable dimension, and A_t, B_t, C_t, D_t are the matrices of the state-space realization; they depend on t because they are functions of $\hat{\theta}_t$. Please don't bother, for the moment, if I keep the zero as a “fake” first component of the input: it is there for future use.

Fact 0.2.2

It is not true, in general, that if the system matrices $A_t = A(\hat{\theta}_t)$ have eigenvalues inside the unit circle, as the controller manages to attain, the time-varying system

$$\xi(t+1) = A_t\ \xi(t)$$

is asymptotically stable. However, it is true in this case because A_t has an “asymptotically stable” limit:

$$\lim_{t\to\infty} A_t = \lim_{t\to\infty} A(\hat{\theta}_t) = A(\hat{\theta}_\infty) = A_\infty.$$

(Recall the assumption: the system frozen at infinity is internally stable.)

Moreover, a *stronger* notion holds because of linearity. The above system is exponentially stable, which means there exist constants $c > 0$ and ν, $0 < \nu < 1$, such that

$$\|\xi(t)\| \le c \cdot \nu^t\ \|\xi(0)\| \qquad \text{for all } t.$$

Take-home message 0.2.3

So, asymptotically, the imaginary systems behaves like the system frozen at infinity. From the above fact we can conclude:

- the imaginary system $\Sigma(\hat{\theta}_t, \hat{\theta}_t)$ in (1) is BIBO-stable (because it is *asymptotically* stable in state-space form);
- the signals $y_{\mathrm{im}}(t), u_{\mathrm{im}}(t)$ are *bounded* (because the system is BIBO and the input $r^*(t)$ is bounded);
- by “continuity” of the design procedure, $\Sigma(\hat{\theta}_t, \hat{\theta}_t)$ also attains reference tracking (e.g. for constant references and sinusoids, that are easily tracked bounded signals):

$$\lim_{t\to\infty} y_{\mathrm{im}}(t) - y_\infty(t) = 0.$$

0.3 The real system behaves like the imaginary system

It is now time to cope with the real system $\Sigma(\hat{\theta}_t, \bar{\theta})$, that is the “true” plant controlled by the self-tuning regulator.

We carry on the same analysis as before; you will spot immediately the difference:

$$\Sigma(\hat{\theta}_t, \bar{\theta}) : \begin{cases} \alpha(z^{-1}; \bar{\theta}) y(t) = \beta(z^{-1}; \bar{\theta}) u(t) \\ \pi(z^{-1}; \hat{\theta}_t) y(t) = \rho(z^{-1}; \hat{\theta}_t) (\underbrace{r(t) - y(t)}_{e(t)}) \end{cases}$$

$$
\Sigma(\hat{\theta}_t, \bar{\theta}) : \begin{cases} y(t) = (1 - \alpha(z^{-1}; \bar{\theta}))y(t) + \beta(z^{-1}; \bar{\theta})u(t) \\ \quad\;\;\; = \bar{a}_1 y(t-1) + \bar{a}_2 y(t-2) + \bar{a}_3 y(t-3) + \bar{b}_1 u(t-1) + \bar{b}_2 u(t-2) + \bar{b}_3 u(t-3) \\ \quad\;\;\; = \varphi_t^\top \bar{\theta} \\ \quad\;\;\; = \varphi_t^\top (\bar{\theta} - \hat{\theta}_t + \hat{\theta}_t) \\ \quad\;\;\; = \varphi_t^\top \hat{\theta}_t + \underbrace{\left(-\varphi_t^\top \tilde{\theta}_t\right)}_{:=\varepsilon(t)} \\ u(t) = (1 - \pi(z^{-1}; \hat{\theta}_t))u(t) - \rho(z^{-1}; \hat{\theta}_t)y(t) + \underbrace{\rho(z^{-1}; \hat{\theta}_t)r(t)}_{:=r^*(t)} \end{cases}
$$

$$
\Sigma(\hat{\theta}_t, \bar{\theta}) : \begin{cases} \xi(t+1) = A_t \xi(t) + B_t \begin{bmatrix} \varepsilon(t) \\ r^*(t) \end{bmatrix} \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = C_t \xi(t) + D_t \begin{bmatrix} \varepsilon(t) \\ r^*(t) \end{bmatrix} \end{cases} \tag{2}
$$

The only difference with the imaginary system is the presence, in (2), of the input $\varepsilon(t) := -\varphi_t^\top \tilde{\theta}_t$; but it is a substantial difference, because it makes the system *nonlinear* (the matrices of the system do depend on $\tilde{\theta}_t$). If we manage to prove that

- $\Sigma(\hat{\theta}_t, \bar{\theta})$ is BIBO-stable, and
- $\lim_{t\to\infty} \varepsilon(t) = 0$,

then we are in good shape, because if $\Sigma(\hat{\theta}_t, \bar{\theta})$ is BIBO-stable the effect of $\varepsilon(t)$ on $y(t), u(t)$ is just a transient.

Resuming: if we prove that $\Sigma(\hat{\theta}_t, \bar{\theta})$ is BIBO-stable, and $\varepsilon(t) \to 0$, then

$$
\begin{aligned}
\lim_{t\to\infty} y(t) - r(t) &= \lim_{t\to\infty} \underbrace{(y(t) - y_{\text{im}}(t))}_{\Sigma(\hat{\theta}_t, \bar{\theta}) \text{ behaves like } \Sigma(\hat{\theta}_t, \hat{\theta}_t)} + \underbrace{(y_{\text{im}}(t) - y_\infty(t))}_{\Sigma(\hat{\theta}_t, \hat{\theta}_t) \text{ behaves like } \Sigma(\hat{\theta}_\infty, \hat{\theta}_\infty)} + \underbrace{(y_\infty(t) - r(t))}_{\Sigma(\hat{\theta}_\infty, \hat{\theta}_\infty) \text{ enforces reference tracking}} \\
&= 0,
\end{aligned}
$$

so that the real system also enforces reference tracking.

0.4 The real system is BIBO-stable and $\varepsilon(t) \to 0$

For a starter, note that $\varepsilon(t) = -\varphi_t^\top \tilde{\theta}_t$ is bounded. Indeed, we know from the convergence analysis of the LS estimate that

$$
\left(\varphi_t^\top \tilde{\theta}_t\right)^2 = \tilde{\theta}_t^\top \left(\varphi_t \varphi_t^\top\right) \tilde{\theta}_t \leq \tilde{\theta}_t^\top \left(\lambda I + \sum_{\tau=0}^{t} \varphi_\tau \varphi_\tau^\top\right) \tilde{\theta}_t \leq \tilde{\theta}_t^\top R_t \tilde{\theta}_t \leq V_0,
$$

hence $|\varepsilon(t)| \leq \sqrt{V_0}$.

Now recall that the autonomous system $\xi(t+1) = A_t \xi(t)$ is not just *asymptotically* stable, but *exponentially* stable: there exist positive constants c, $\nu < 1$ such that $\|\xi(t)\| \leq c \cdot \nu^t \|\xi(0)\|$ for all t.

Keep in mind the following elementary

Fact 0.4.1

If $0 < \nu < 1$ and the sequence of numbers $s(t)$ is bounded, then the sequence

$$S(t) = \sum_{\tau=0}^{t} \nu^{t-\tau} s(\tau)$$

is also bounded. If, moreover, $s(t) \to 0$ as $t \to \infty$, then also $S(t) \to 0$.

Indeed, the above is the response of the asymptotically stable scalar system $x(t+1) = \nu x(t) + s(t)$ to a bounded input.

For brevity, denote

$$v(t) := \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}.$$

Expanding the recursion from the state-space system (2), we obtain:

$$\begin{aligned}
v(t) &= C_t \xi(t) + D_t \begin{bmatrix} \varepsilon(t) \\ r^*(t) \end{bmatrix} \\
&= C_t \underbrace{A_{t-1}}_{\text{dominated by } \nu} \xi(t-1) + C_t B_{t-1} \begin{bmatrix} \varepsilon(t-1) \\ r^*(t-1) \end{bmatrix} + D_t \begin{bmatrix} \varepsilon(t) \\ r^*(t) \end{bmatrix} \\
&= C_t \underbrace{A_{t-1} A_{t-2}}_{\text{dominated by } \nu^2} \xi(t-2) + C_t \underbrace{A_{t-1}}_{\text{dominated by } \nu} B_{t-2} \begin{bmatrix} \varepsilon(t-2) \\ r^*(t-2) \end{bmatrix} + C_t B_{t-1} \begin{bmatrix} \varepsilon(t-1) \\ r^*(t-1) \end{bmatrix} + D_t \begin{bmatrix} \varepsilon(t) \\ r^*(t) \end{bmatrix} \\
&= C_t \underbrace{A_{t-1} A_{t-2} A_{t-3}}_{\text{dominated by } \nu^3} \xi(t-3) + \dots
\end{aligned}$$

and so on. Omitting pointless details, it follows that there exist constants k_1, k_2 such that

$$\begin{aligned}
\|v(t)\| &\leq k_1 + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} \left\| \begin{bmatrix} \varepsilon(\tau) \\ r^*(\tau) \end{bmatrix} \right\| \\
&\leq k_1 + k_2 \underbrace{\sum_{\tau=0}^{t} \nu^{t-\tau} |\varepsilon(\tau)|}_{\text{bounded}} + k_2 \underbrace{\sum_{\tau=0}^{t} \nu^{t-\tau} |r^*(\tau)|}_{\text{bounded}}.
\end{aligned}$$

Therefore, $v(t)$ is bounded (and so are its components $y(t)$ and $u(t)$).

Let's expand further: let $\tilde{\theta}_t = \tilde{\theta}_t^{\mathcal{E}} + \tilde{\theta}_t^{\mathcal{U}}$ be the orthogonal decomposition of $\tilde{\theta}_t$ along the excitation and unexcitation subspaces:

$$\begin{aligned}
\|v(t)\| &\leq k_1 + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |\varphi_\tau^\top (\tilde{\theta}_\tau^{\mathcal{E}} + \tilde{\theta}_\tau^{\mathcal{U}})| + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |r^*(\tau)| \\
&\leq k_1 + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |\varphi_\tau^\top \tilde{\theta}_\tau^{\mathcal{E}}| + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |\varphi_\tau^\top \tilde{\theta}_\tau^{\mathcal{U}}| + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |r^*(\tau)|
\end{aligned}$$

Leave the sum containing $\tilde{\theta}_\tau^{\mathcal{E}}$ explicit and dominate everything else with a constant k_3:

$$\begin{aligned}
\|v(t)\| &\leq k_3 + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} |\varphi_\tau^\top \tilde{\theta}_\tau^{\mathcal{E}}| \\
&\leq k_3 + k_2 \sum_{\tau=0}^{t} \nu^{t-\tau} \|\varphi_\tau\| \cdot \left\| \tilde{\theta}_\tau^{\mathcal{E}} \right\| \qquad \text{(Cauchy-Schwarz inequality).}
\end{aligned} \tag{3}$$

Now consider:

$$\|\varphi_t\| = \left\| \begin{bmatrix} y(t-1) \\ y(t-2) \\ y(t-3) \\ u(t-1) \\ u(t-2) \\ u(t-3) \end{bmatrix} \right\| = \left\| \begin{bmatrix} y(t-1) \\ u(t-1) \\ y(t-2) \\ u(t-2) \\ y(t-3) \\ u(t-3) \end{bmatrix} \right\| = \left\| \begin{bmatrix} v(t-1) \\ v(t-2) \\ v(t-3) \end{bmatrix} \right\|$$
$$\leq \|v(t-1)\| + \|v(t-2)\| + \|v(t-3)\| .$$

(The generalization to ARX(m, n) models is obvious.) Each term in the last sum is dominated exponentially (according to (3)) by terms <u>up to time $t-1$</u>; therefore, omitting pointless details, there exist constants k_4, k_5 such that

$$\begin{aligned} \|\varphi_t\| &\leq k_4 + k_5 \sum_{\tau=0}^{t-1} \nu^{t-\tau} \|\varphi_\tau\| \cdot \left\|\tilde{\theta}_\tau^{\mathcal{E}}\right\| \\ &\leq k_4 + \left(\max_{\tau=0,\ldots,t-1} \|\varphi_\tau\| \right) \cdot \underbrace{k_5 \sum_{\tau=0}^{t-1} \nu^{t-\tau} \left\|\tilde{\theta}_\tau^{\mathcal{E}}\right\|}_{\text{tends to 0 as } t \to \infty} . \end{aligned} \tag{4}$$

I promise that we are close to the end.

Fact 0.4.2 (A technical lemma)

Suppose that that the sequences $x_t \geq 0$ and $u_t \geq 0$ satisfy the following inequality,

$$x_t \leq a + \left(\max_{\tau=0,\ldots,t-1} x_\tau \right) u_t,$$

where $a > 0$ is a constant, and suppose that $\lim_{t\to\infty} u_t = 0$.

Then the sequence $\{x_t\}$ is bounded.

To prove the technical lemma assume, for the sake of contradiction, that $\{x_t\}$ is *not* bounded. Then there exists an index T such that:

$$\begin{aligned} \max_{\tau=0,\ldots,T} x_\tau &\geq 2a, \\ u_t &\leq 1/2 \qquad \text{for all } t \geq T. \end{aligned}$$

Define $C := \max_{\tau=0,\ldots,T} x_\tau$ (so that $a \leq C/2$). It follows:

$$\begin{aligned} x_{T+1} &\leq \frac{C}{2} + \left(\max_{\tau=0,\ldots,T} x_\tau \right) \cdot \frac{1}{2} \leq \frac{C}{2} + \frac{C}{2} = C; \\ x_{T+2} &\leq \frac{C}{2} + \left(\max_{\tau=0,\ldots,T+1} x_\tau \right) \cdot \frac{1}{2} \qquad \text{(because also } x_{T+1} \leq C) \\ &\leq \frac{C}{2} + \frac{C}{2} = C; \\ x_{T+3} &\leq \frac{C}{2} + \left(\max_{\tau=0,\ldots,T+2} x_\tau \right) \cdot \frac{1}{2} \leq C, \end{aligned}$$

and so on. This contradicts the assumption (indeed $x_t \leq C$ for all t, but it was assumed to be unbounded), hence the sequence is bounded. □

Applying the lemma to equation (4) with $x_t = \|\varphi_t\|$, $a = k_4$, and $u_t = k_5 \sum_{\tau=0}^{t-1} \nu^{t-\tau} \left\|\tilde{\theta}_\tau^{\mathcal{E}}\right\|$, we obtain: <u>$\|\varphi_t\|$ is bounded</u>.

We can now prove that $\varepsilon(t) \to 0$. We start with the component $\tilde{\theta}_t^{\mathcal{U}}$ along the unexcitation subspace; note that

$$\varphi_t^\top \tilde{\theta}_t^{\mathcal{U}} = \underbrace{\varphi_t^\top}_{\text{bounded}} \underbrace{\left(\tilde{\theta}_t^{\mathcal{U}} - \tilde{\theta}_\infty^{\mathcal{U}}\right)}_{\text{converges to } 0} + \varphi_t^\top \tilde{\theta}_\infty^{\mathcal{U}};$$

since the first term on the right-hand side converges to 0, the limits of $\varphi_t^\top \tilde{\theta}_t^{\mathcal{U}}$ and $\varphi_t^\top \tilde{\theta}_\infty^{\mathcal{U}}$ must coincide. To obtain the second one, recall that

$$\begin{aligned}\lim_{t\to\infty} \sum_{\tau=0}^{t} \left(\varphi_\tau^\top \tilde{\theta}_\infty^{\mathcal{U}}\right)^2 &= \lim_{t\to\infty} (\tilde{\theta}_\infty^{\mathcal{U}})^\top \left(\sum_{\tau=0}^{t} \varphi_\tau \varphi_\tau^\top\right) \tilde{\theta}_\infty^{\mathcal{U}} \\ &\le \lim_{t\to\infty} (\tilde{\theta}_\infty^{\mathcal{U}})^\top \; R_t \; \tilde{\theta}_\infty^{\mathcal{U}} \\ &< +\infty\end{aligned}$$

because $\tilde{\theta}_\infty^{\mathcal{U}} \in \mathcal{U}$. In other terms the series $\sum_{\tau=0}^{\infty} \left(\varphi_\tau^\top \tilde{\theta}_\infty^{\mathcal{U}}\right)^2$ converges; for this to happen its term must tend to 0. Therefore we have

$$\lim_{t\to\infty} \varphi_t^\top \tilde{\theta}_t^{\mathcal{U}} = \lim_{t\to\infty} \varphi_t^\top \tilde{\theta}_\infty^{\mathcal{U}} = 0.$$

Note: this means that in the long run φ_τ is always "almost" *orthogonal* to $\tilde{\theta}_\infty^{\mathcal{U}}$; it does <u>not</u> mean that $\tilde{\theta}_\infty^{\mathcal{U}} = 0$. Remember: *only in the excitation subspace* the component of the estimation error tends to 0.

Finally, apply the decomposition $\tilde{\theta}_t = \tilde{\theta}_t^{\mathcal{E}} + \tilde{\theta}_t^{\mathcal{U}}$ to $\varepsilon(t)$:

$$\begin{aligned}\lim_{t\to\infty} |\varepsilon(t)| &= \lim_{t\to\infty} \left|\varphi_t^\top \tilde{\theta}_t\right| \\ &\le \lim_{t\to\infty} \left|\varphi_t^\top \tilde{\theta}_t^{\mathcal{E}}\right| + \left|\varphi_t^\top \tilde{\theta}_t^{\mathcal{U}}\right| \\ &\le \lim_{t\to\infty} \underbrace{\|\varphi_t\|}_{\text{bounded}} \cdot \underbrace{\left\|\tilde{\theta}_t^{\mathcal{E}}\right\|}_{\to 0} + \underbrace{\left|\varphi_t^\top \tilde{\theta}_t^{\mathcal{U}}\right|}_{\to 0} \\ &= 0.\end{aligned}$$

0.5 Conclusion

Take-home message 0.5.1

Looking at the closed loop from outside, $r(t)$ is its only input, and $u(t), y(t)$ are its outputs. We have shown that, if $r(t)$ is assumed to be bounded, then the vector $v(t) = (y(t), u(t))$ is bounded, and such are, of course, its components $y(t)$ and $u(t)$. Now, this is the definition of BIBO-stability; hence,

<u>The closed loop/real system $\Sigma(\hat{\theta}_t, \bar{\theta})$ is BIBO-stable.</u>

We have also taken a long tour to show that $\varepsilon(t) \to 0$ (it is a transient perturbation of the imaginary system); but in the end this allows to obtain:

$$\lim_{t\to\infty} y(t) - r(t) = \lim_{t\to\infty} \underbrace{(y(t) - y_{\text{im}}(t))}_{\Sigma(\hat{\theta}_t,\bar{\theta}) \sim \Sigma(\hat{\theta}_t,\hat{\theta}_t)} + \underbrace{(y_{\text{im}}(t) - y_\infty(t))}_{\Sigma(\hat{\theta}_t,\hat{\theta}_t) \sim \Sigma(\hat{\theta}_\infty,\hat{\theta}_\infty)} + \underbrace{(y_\infty(t) - r(t))}_{\Sigma(\hat{\theta}_\infty,\hat{\theta}_\infty) \text{ enforces reference tracking}} = 0,$$

that is,

<u>The real system $\Sigma(\hat{\theta}_t, \bar{\theta})$ also enforces reference tracking.</u>