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On mean square boundedness of stochastic linear systems with bounded controls[☆]

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ABSTRACT

We start by summarizing the state of the art in stabilization of stochastic linear systems with bounded inputs and highlight remaining open problems. We then report two new results concerning mean-square boundedness of a linear system with additive stochastic noise. The first states that, given any nonzero bound on the controls, it is possible to construct a policy with bounded memory requirements that renders a marginally stable stabilizable system mean-square bounded in closed-loop. The second states that it is not possible to ensure mean-square boundedness in closed-loop with a bounded control policy for systems affected by unbounded noise and having at least one eigenvalue outside the unit circle.

1. Introduction and results

This article concerns a problem of stability of stochastic linear controlled systems

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 \text{ given}, \ t \in \mathbb{N}_0, \tag{1.1}$$

where x_t, u_t are the state and the control vectors at time t, respectively, $(w_t)_{t\in\mathbb{N}_0}$ is the (not necessarily bounded) mean-zero noise with bounded variance, assumed i.i.d. for the moment, $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$. We are interested in determining a suitable control sequence $(u_t)_{t\in\mathbb{N}_0}$ under which the state sequence $(x_t)_{t\in\mathbb{N}_0}$ is mean-square bounded. To wit, our objective is to determine, if possible, a controller satisfying $\|u_t\| \leqslant R$ for all t and $\gamma > 0$ such that $\sup_{t\in\mathbb{N}_0} \mathsf{E}[\|x_t\|^2] \leqslant \gamma$.[1] The crux of the matter is the presence of the bound on the control actions. Even in the deterministic noise-free setting, global asymptotic stabilization of the system is a rather non-trivial matter—see e.g., [1,2] for details. (Analogous to the deterministic setting, one could potentially think of establishing conditions in the stochastic context that ensure $\mathsf{E}_{x_0}[\|x_t\|^2] \xrightarrow[t\to\infty]{} 0$; this, however, necessarily requires that the variance of the noise in (1.1) vanishes asymptotically. Here we shall stick to the standard assumption of non-vanishing variance of the noise.) In the stochastic setting the interplay of spectral radius and Lyapunov stability of the matrix A gives rise to four distinct cases, stated next.

1.1. A is Schur stable

If the matrix A is Schur stable, (i.e., the eigenvalues of A are all inside the open unit disc,) standard Foster–Lyapunov techniques [3] reveal that a bound γ exists whenever the variance of the noise is bounded.

1.2. A is Lyapunov stable

Recall that Lyapunov stability of A implies that its eigenvalues have magnitude at most one 1 and those on the unit circle have equal geometric and algebraic multiplicities. This case was recently treated in [4], where a bounded k-history-dependent policy was constructed which renders the closed-loop system mean-square bounded provided that the control bound is large enough. It is not difficult to see that in this case the system decomposes to a Schur stable part and a part with an orthogonal A matrix. The integer k is then the reachability index of the orthogonal part of the system,

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[1] The choice of the Euclidean norm here is without loss of generality.

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and the control bound required is proportional to the first absolute moment of the noise. This policy is a feedback for the k-subsampled system, and is therefore not a pure feedback for the original system. Nonetheless, the bounded memory requirement of this policy – at most k past states need to be retained at any time – ensures that it is applicable in practical situations. For instance, the policy has been constructively employed in receding horizon control [5] and networked control systems [6] to ensure good qualitative behavior of the closed-loop systems. However, the requirement that the control bound is greater than the first absolute moment of the noise is undesirable and appears to be unnecessary.

Our first result extends the main result of [4] by removing the lower bound on the control authority required there to ensure bounded variance of the closed-loop system. We construct a control policy that ensures that a linear system (1.1) is mean-square bounded in closed-loop when A is Lyapunov stable, the pair (A, B) is reachable in κ steps with arbitrary controls, and $\|u_t\| \leqslant R$ for arbitrary and pre-assigned $R > 0$. Our policy belongs to the class of κ-history-dependent non-stationary policies. We refer the reader to our earlier article [4] for the basic setup, various definitions, and in particular to [4, Section 3.4] for the details about a change of basis in $\mathbb{R}^d$ that shows that it is sufficient to consider A orthogonal. We have the following theorem:

Theorem 1.2. *Consider the system (1.1). Suppose that the pair (A, B) is κ-steps reachable, and that A is orthogonal. In addition, suppose that $(w_t)_{t\in\mathbb{N}_0}$ is a mutually independent $\mathbb{R}^d$-valued mean-zero stochastic noise, and that there exists some $C_4 > 0$ such that $\mathsf{E}\big[\|w_t\|^4\big] \leqslant C_4$ for every t. Let $R > 0$ be given. Then there exists a time-varying κ-history-dependent policy $(\pi_t)_{t\in\mathbb{N}_0}$ with $\kappa \leqslant d$ and $\|\pi_t(\cdot)\| \leqslant R$ for every t, such that for every initial state $x_0 \in \mathbb{R}^d$ there exists $\gamma = \gamma(x_0, R, C_4) > 0$ with the closed-loop system satisfying $\sup_{t\in\mathbb{N}_0} \mathsf{E}_{x_0}\big[\|x_t\|^2\big] \leqslant \gamma$.*

We give a quantitative estimate of the constant γ in Remark 2.7. Theorem 1.2, which yields a time-varying κ-history-dependent policy, also leads to the following natural open question:

Open problem 1.3. *Is it possible to attain bounded closed-loop variance with bounded controls using static state feedback?*

We conjecture that the answer to Problem 1.3 is "yes", but contend that a proof of this will require new techniques.

1.3. A has spectral radius 1 but is not Lyapunov stable

In the deterministic setting with no noise, i.e., the system being $x_{t+1} = Ax_t + Bu_t$, the main result of [2] established that the system can be globally asymptotically stabilized if the pair (A, B) is stabilizable and the spectral radius of A is no more than 1. In the stochastic setting results in this direction were reported in [7], but a conclusive treatment remains elusive.

Open problem 1.4. *Is it possible to attain bounded closed-loop variance with bounded controls for systems with spectral radius at most 1?*

Our proof of Theorem 1.2 exploits orthogonality of the matrix A crucially, consequently, our arguments do not extend in a straightforward fashion to the case of A having non-trivial Jordan blocks. We contend that the settlement of this question will also require new analysis techniques.

1.4. A has spectral radius greater than 1

While it is intuitively reasonable and has been argued, e.g., in [8], that a bound γ on the closed-loop variance does not exist if even one eigenvalue of A has magnitude greater than 1, (see also [9] for an interesting argument in the scalar case,) to the best of our knowledge a general proof in the multi-dimensional case is not available. Our second result concerns this case, and establishes that a linear system with at least one unstable eigenvalue and subjected to unbounded noise cannot be stabilized by means of bounded controls. Let $A \in \mathbb{R}^{d\times d}$, and suppose that λ is a real eigenvalue. Then λ will appear in at least one diagonal block of the real Jordan form of A, for example:

$$\bar{A} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{1.5}$$

With loose terminology, we will call a "last generalized eigenspace relative to λ" a space generated by what in the Jordan basis is the canonical vector corresponding to the last column of a block (e.g., either e_3 or e_5, if we refer to (1.5)). In the same fashion, suppose that $\sigma \pm \mathrm{i}\omega$ is a conjugate pair of eigenvalues. Then in the *real* Jordan form of A that pair will appear in the form of 2×2 blocks, for example as follows:

$$\bar{A} = \begin{bmatrix} \sigma & -\omega & 1 & 0 & 0 & 0 & 0 & 0 \\ \omega & \sigma & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & -\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{1.6}$$

and we will call a "last generalized eigenspace relative to $\sigma \pm \mathrm{i}\omega$" a space generated by the *two* vectors that in the real Jordan basis read as the canonical vectors corresponding to the last two columns of a block (e.g., $\mathrm{Span}\{e_3, e_4\}$ with respect to (1.6)).

Theorem 1.7. *Consider the linear system (1.1), where $A \in \mathbb{R}^{d\times d}$ has at least one eigenvalue λ with $|\lambda| > 1$, $u_t \in \mathbb{R}^m$ is a bounded control action for any time t, and $(w_t)_{t\in\mathbb{N}_0}$ is an i.i.d. d-dimensional random noise independent of x_0. Let ν_t be the projection of w_t on a last generalized eigenspace relative to the unstable eigenvalue (or unstable conjugate pair) of A, and suppose that $(\nu_t)_{t\in\mathbb{N}_0}$ has unbounded support, that is, $\mathsf{P}\big(\|\nu_t\| > M\big) > 0$ for all $M > 0$ and $t \in \mathbb{N}_0$. Then for any $x_0 \in \mathbb{R}^d$, $\sup_{t\in\mathbb{N}_0} \mathsf{E}\big[\|x_t\|\big] = \infty$, and, consequently, $\sup_{t\in\mathbb{N}_0} \mathsf{E}\big[\|x_t\|^2\big] = \infty$.*

The proofs of Theorems 1.2 and 1.7 are provided in Section 2.

2. Proofs

For $r > 0$ the standard r-saturation function is $\mathbb{R} \ni y \longmapsto \mathrm{sat}_r(y) := \mathrm{sgn}(y) \times \min\{r, |y|\} \in [-r, r]$. For a matrix $M \in \mathbb{R}^{m\times n}$ we let $\sigma_1(M)$ denote its maximal singular value and M^+ its Moore–Penrose pseudo-inverse. For a vector $v \in \mathbb{R}^n$ we let $v^{(k)}$ denote its k-th entry, $k = 1, \ldots, n$. For a real-valued random variable X on some probability space, we let $X^+ := \max\{0, X\}$, $X^- := \max\{0, -X\}$ denote its positive and negative parts, respectively. For a stochastic process $(X_t)_{t\in\mathbb{N}_0}$ defined on some probability space and f a measurable function, we denote by $\mathsf{E}_x[f(X_t)] := \mathsf{E}[f(X_t) \mid X_0 = x]$ the conditional expectation of $f(X_t)$ given initial condition $X_0 = x$.

We need the following basic result derived from [10]:

Proposition 2.1. *Let $(\xi_t)_{t\in\mathbb{N}_0}$ be a sequence of scalar random variables on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, and let $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$ be any filtration to which $(\xi_t)_{t\in\mathbb{N}_0}$ is adapted. Suppose that there exist* *constants $a > 0$, and $J, M < \infty$, such that $\xi_0 \leqslant J$, and for all t:*

$$\mathsf{E}^{\mathfrak{F}_t}[\xi_{t+1} - \xi_t] \leqslant -a \quad \text{on the event } \{\xi_t > J\}, \tag{2.2}$$

and

$$\mathsf{E}\left[|\xi_{t+1} - \xi_t|^4 \,\middle|\, \xi_0, \ldots, \xi_t\right] \leqslant M. \tag{2.3}$$

Then there exists a constant $c = c(a, J, M) > 0$ such that $\sup_{t\in\mathbb{N}_0} \mathsf{E}\big[(\xi_t^+)^2\big] \leqslant c$.

Proof of Theorem 1.2. Fix $R > 0$. Consider the κ-sub-sampled version of (1.1):

$$\begin{aligned} x_{\kappa(t+1)} &= A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\begin{bmatrix} u_{\kappa t} \\ \vdots \\ u_{\kappa(t+1)-1}\end{bmatrix} \\ &\quad + \mathcal{R}_\kappa(A,I)\begin{bmatrix} w_{\kappa t} \\ \vdots \\ w_{\kappa(t+1)-1}\end{bmatrix} \\ &=: A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_t + \bar{w}_t, \quad t \in \mathbb{N}_0, \end{aligned} \tag{2.4}$$

with initial condition x_0 given, $\mathcal{R}_k(A, M) := \big[A^{k-1}M \;\cdots\; AM\; M\big]$ for any matrix M with d rows, and I the $d \times d$ identity matrix. Observe that $\mathcal{R}_\kappa(A, B)$ has full rank due to the pair (A, B) being κ-step reachable. Here $\bar{u}_t \in \mathbb{R}^{\kappa m}$, with each block component being $u_{\kappa t+\ell}$ with $\|u_{\kappa t+\ell}\| \leqslant R$, $\ell = 0, 1, \ldots, \kappa - 1$, and $(\bar{w}_t)_{t\in\mathbb{N}_0}$ is $\mathbb{R}^d$-valued mean-zero stochastic noise with $\sup_{t\in\mathbb{N}_0} \mathsf{E}\big[\|\bar{w}_t\|^4\big] \leqslant \sigma_1\big(\mathcal{R}_\kappa(A, I)\big)^4 C_4$.

Defining $y_t := \big(A^{\kappa t}\big)^\top x_{\kappa t} \in \mathbb{R}^d$, in view of A being orthogonal, we get

$$\begin{aligned} y_{t+1} &= \big(A^{\kappa(t+1)}\big)^\top x_{\kappa(t+1)} \\ &= \big(A^{\kappa(t+1)}\big)^\top \big(A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_t + \bar{w}_t\big) \\ &= y_t + \widetilde{u}_t + \widetilde{w}_t, \end{aligned} \tag{2.5}$$

where $\widetilde{u}_t = (A^\top)^{\kappa(t+1)}\mathcal{R}_\kappa(A, B)\bar{u}_t$, $\widetilde{w}_t = (A^\top)^{\kappa(t+1)}\bar{w}_t$, with the sequence $(\widetilde{w}_t)_{t\in\mathbb{N}_0}$ being $\mathbb{R}^d$-valued mean-zero noise with $\sup_{t\in\mathbb{N}_0} \mathsf{E}\big[\|\widetilde{w}_t\|^4\big] \leqslant \sigma_1\big(\mathcal{R}_\kappa(A, I)\big)^4 C_4$. Trivially, therefore, each $\big(\widetilde{w}_t^{(k)}\big)_{t\in\mathbb{N}_0}$ is mean-zero and fourth-moment bounded. Selecting $r \in \left]0, \frac{R}{(\sqrt{d}\sigma_1(\mathcal{R}_\kappa(A,B)^+))}\right[$, we define $\widetilde{u}_t^{(k)} := -\mathrm{sat}_r\big(y_t^{(k)}\big)$, $k = 1, \ldots, d$. Let $\mathfrak{F}_t$ be the σ-algebra generated by the random vectors $\{x_{\kappa s}\}_{s=0}^t$; note that $(y_t)_{t\in\mathbb{N}_0}$ is $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$-adapted.

Since $\big|y_t^{(k)}\big| = \big(y_t^{(k)}\big)^+ + \big(y_t^{(k)}\big)^- = \big(y_t^{(k)}\big)^+ + \big(-y_t^{(k)}\big)^+$, a trivial application of the triangle inequality shows that $\left(\mathsf{E}_{y_0}\big[\big|y_t^{(k)}\big|^2\big]\right)_{t\in\mathbb{N}_0}$ is bounded provided both the sequences $\left(\mathsf{E}_{y_0}\Big[\big(\big(y_t^{(k)}\big)^+\big)^2\Big]\right)_{t\in\mathbb{N}_0}$ and $\left(\mathsf{E}_{y_0}\Big[\big(\big(y_t^{(k)}\big)^-\big)^2\Big]\right)_{t\in\mathbb{N}_0}$ are bounded. To this end, consider the dynamics of the k-th component of $(y_t)_{t\in\mathbb{N}_0}$:

$$y_{t+1}^{(k)} = y_t^{(k)} - \mathrm{sat}_r\big(y_t^{(k)}\big) + \widetilde{w}_t^{(k)}, \qquad y_0^{(k)} \text{ given}, \quad t \in \mathbb{N}_0. \tag{2.6}$$

In view of the above arguments, observe that

$$\mathsf{E}^{\mathfrak{F}_t}\big[y_{t+1}^{(k)} - y_t^{(k)}\big] = -\mathrm{sat}_r\big(y_t^{(k)}\big) = -r \quad \text{on the set} \big\{y_t^{(k)} > r\big\}.$$

Moreover, the triangle inequality gives

$$\mathsf{E}\Big[\big|y_{t+1}^{(k)} - y_t^{(k)}\big|^4 \,\Big|\, \big\{y_s^{(k)}\big\}_{s=0}^t\Big] \leqslant \Big|\mathsf{E}\Big[\big(r + \big|\widetilde{w}_t^{(k)}\big|\big)^4 \,\Big|\, \big\{y_s^{(k)}\big\}_{s=0}^t\Big] \leqslant M$$

for some $M = M(r, C_4) > 0$. We take $J_+ = \max\{(y_0^{(k)}), r\}$ and verify that the conditions of Proposition 2.1 hold for the sequence $\big(y_t^{(k)}\big)_{t\in\mathbb{N}_0}$. This shows that there exists a constant $\gamma_+^{(k)} = \gamma_+^{(k)}(y_0, R, C_4) > 0$ such that $\mathsf{E}_{y_0}\Big[\big(\big(y_t^{(k)}\big)^+\big)^2\Big] \leqslant \gamma_+^{(k)}$ for all t. An identical calculation with $\big(-y_t^{(k)}\big)_{t\in\mathbb{N}_0}$ instead of $\big(y_t^{(k)}\big)_{t\in\mathbb{N}_0}$ shows that there exists a constant $\gamma_-^{(k)} = \gamma_-^{(k)}(y_0, R, C_4) > 0$ such that $\mathsf{E}_{y_0}\Big[\big(\big(y_t^{(k)}\big)^-\big)^2\Big] \leqslant \gamma_-^{(k)}$. In view of the discussion above, $\mathsf{E}_{y_0}\big[\big|y_t^{(k)}\big|^2\big] \leqslant 2\big(\gamma_+^{(k)} + \gamma_-^{(k)}\big)$ for all $t \in \mathbb{N}_0$. Furthermore, since A is orthogonal, $\|x_{\kappa t}\| = \|y_t\|$ and $x_0 = y_0$; thus,

$$\mathsf{E}_{x_0}\big[\|x_{\kappa t}\|^2\big] \leqslant \sum_{k=1}^d \left(\gamma_+^{(k)} + \gamma_-^{(k)}\right) =: \gamma \quad \text{for all } t.$$

A standard argument, e.g., as in [4, Proof of Lemma 3.14] shows that $(x_t)_{t\in\mathbb{N}_0}$ is mean-square bounded if $(x_{\kappa t})_{t\in\mathbb{N}_0}$ is; therefore there exists a constant γ depending on (x_0, R, C_4) (and, of course, the system parameters,) such that $\mathsf{E}_{x_0}\big[\|x_t\|^2\big] \leqslant \gamma$ for all $t \in \mathbb{N}_0$.

We recover the controls $(u_t)_{t\in\mathbb{N}_0}$ by noting that $\mathcal{R}_\kappa(A, B)$ has rank d, hence $(A^\top)^{\kappa(t+1)}\mathcal{R}_\kappa(A, B)$ is left-invertible; so, for $t \in \mathbb{N}_0$,

$$\widetilde{u}_t^{(k)} = -\mathrm{sat}_r\big(y_t^{(k)}\big) = -\mathrm{sat}_r\Big(\big((A^\top)^{\kappa t}x_{\kappa t}\big)^{(k)}\Big) \in \mathbb{R},$$

$$k = 1, \ldots, d,$$

$$\begin{aligned} \bar{u}_t &= \begin{bmatrix} u_{\kappa t} \\ \vdots \\ u_{\kappa(t+1)-1}\end{bmatrix} = \big((A^\top)^{\kappa(t+1)}\mathcal{R}_\kappa(A,B)\big)^+\widetilde{u}_t \\ &= \mathcal{R}_\kappa(A,B)^+ A^{\kappa(t+1)}\widetilde{u}_t \in \mathbb{R}^{\kappa m}, \end{aligned}$$

where the last equality is due to the fact that $\mathcal{R}_\kappa(A, B)$ has full row rank and that A is non-singular. To wit, for each $t \in \mathbb{N}_0$, the control $u_{\kappa t+\ell} \in \mathbb{R}^m$ depends only on $x_{\kappa t}$ for $\ell = 0, \ldots, \kappa - 1$, and is at most κ-history-dependent; and due to the presence of the time-varying factor $(A^\top)^{\kappa(t+1)}$, the policy is time-varying. The controls are bounded (in the Euclidean norm) by R by construction because for $t \in \mathbb{N}_0$ and $\ell \in \{0, 1, \ldots, \kappa - 1\}$,

$$\begin{aligned} \|u_{\kappa t+\ell}\| &\leqslant \|\bar{u}_t\| \leqslant \sigma_1\big(\mathcal{R}_\kappa(A,B)^+\big)\,\|\widetilde{u}_t\| \\ &\leqslant \sigma_1\big(\mathcal{R}_\kappa(A,B)^+\big)\sqrt{d}r \leqslant R. \end{aligned}$$

Since R was arbitrary, the proof is complete. □

Remark 2.7. From the remark following [10, Corollary 2] we get the expression for the bound γ in Theorem 1.2 given below. Referring to the proof of Theorem 1.2 above for the notation, we have $\gamma \leqslant \sum_{k=1}^d \gamma^{(k)} = 2\sum_{k=1}^d \big(\gamma_+^{(k)} + \gamma_-^{(k)}\big)$, where

$$\begin{aligned} \gamma_+^{(k)}, \gamma_-^{(k)} &\leqslant \max\{r, \|x_0\|\} + \pi^2 r^2 2^{14} 3^4 (1 + 3^{-4})^4 \\ &\quad \times \left(918 + \pi^2\left(11 + 3^4\left(2^4\left(1 + \frac{M}{r^4}\right)\right.\right.\right. \\ &\quad \left.\left.\left. + \left(1 + 2^4\left(1 + \frac{M}{r^4}\right)\right)^4\right)\right)\right) \\ &\quad \times \left(2^4\left(1 + \frac{M}{r^4}\right) + \left(1 + 2^4\left(1 + \frac{M}{r^4}\right)\right)^4\right)^2, \end{aligned}$$

and M is bounded above by $8(r^4 + C_4)$. □

Proof of Theorem 1.7. First of all, note that if w_t does not have finite first absolute moment then, for all $t \in \mathbb{N}_0$, x_t does not have finite first absolute moment either, and the claim holds. We may therefore assume that $C_1 := \mathsf{E}[\|w_t\|] < \infty$. The same, of course, holds for any projection ν_t of w_t on a subspace of $\mathbb{R}^d$, i.e., $\mathsf{E}[\|\nu_t\|] =: c_1 < \infty$.

Consider a real change of basis which brings A in its real Jordan form

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t + \bar{u}_t + \bar{w}_t. \tag{2.8}$$

If A has an unstable real eigenvalue λ, without loss of generality we can restrict our attention to the dynamics of the system along a "last generalized eigenspace" relative to λ. In the real Jordan basis, this is the space generated by the canonical vector e_n, where n is the index of the last column of a Jordan block. Referring to the example in (1.5), we can consider the dynamics along $e_n = e_3$; for any $a \in \mathbb{R}$ it holds

$$\bar{A}(ae_3) = ae_2 + (\lambda a)e_3,$$

whose projection onto e_3 is just $(\lambda a)e_3$. Now we let $\xi_t \in \mathbb{R}$, $\upsilon_t \in \mathbb{R}$, and ν_t be the components of $\bar{x}_t$, $\bar{u}_t$, and $\bar{w}_t$ along e_n. Since u_t is bounded, $\bar{u}_t$ is also bounded, hence $\|\upsilon_t\| \leqslant \rho$ for a certain $\rho > 0$. Also, $\mathsf{E}[\|\nu_t\|] \leqslant c_1$ for a certain $c_1 > 0$. In analogy to what happens in the example, the dynamics of ξ_t is given by

$$\xi_{t+1} = \lambda\xi_t + \upsilon_t + \nu_t.$$

Given the component ξ_τ at a certain time τ, we have that

$$\begin{aligned}
\mathsf{E}\big[\|\xi_{t+\tau}\| \,|\xi_\tau\big] &= \mathsf{E}\left[\left\|\lambda^t\xi_\tau + \sum_{k=0}^{t-1}\lambda^{t-1-k}\upsilon_{\tau+k}\right.\right.\\
&\quad \left.\left. + \sum_{k=0}^{t-1}\lambda^{t-1-k}\nu_{\tau+k}\right\| \,\middle|\, \xi_\tau\right]\\
&\geqslant \|\lambda^t\xi_\tau\| - \mathsf{E}\left[\left\|\sum_{k=0}^{t-1}\lambda^{t-1-k}\upsilon_{\tau+k}\right\| \,\middle|\, \xi_\tau\right]\\
&\quad - \mathsf{E}\left[\left\|\sum_{k=0}^{t-1}\lambda^{t-1-k}\nu_{\tau+k}\right\| \,\middle|\, \xi_\tau\right]\\
&\geqslant |\lambda|^t\|\xi_\tau\| - \sum_{k=0}^{t-1}|\lambda|^{t-1-k}\mathsf{E}\big[\|\upsilon_{\tau+k}\| \,\big|\, \xi_\tau\big]\\
&\quad - \sum_{k=0}^{t-1}|\lambda|^{t-1-k}\mathsf{E}\big[\|\nu_{\tau+k}\| \,\big|\, \xi_\tau\big]\\
&\geqslant |\lambda|^t\|\xi_\tau\| - (\rho + c_1)\sum_{k=0}^{t-1}|\lambda|^k\\
&= |\lambda|^t\|\xi_\tau\| - (\rho + c_1)\frac{|\lambda|^t - 1}{|\lambda| - 1}\\
&\geqslant \alpha^t\left(\|\xi_\tau\| - \beta\right),
\end{aligned}$$

where $\alpha = |\lambda|$ and $\beta = \frac{\rho+c_1}{|\lambda|-1}$.

If A has an unstable conjugate pair of eigenvalues $\sigma \pm \mathrm{i}\omega$, we restrict again our attention to a two-dimensional "last generalized eigenspace". Referring to the example in (1.6), we can consider the dynamics along $\mathrm{Span}\{e_3, e_4\}$, for whose generic element it holds

$$\begin{aligned}
\bar{A}(ae_3 + be_4) &= a(e_1 + \sigma e_3 + \omega e_4) + b(e_2 - \omega e_3 + \sigma e_4)\\
&= ae_1 + be_2 + (\sigma a - \omega b)e_3 + (\omega a + \sigma b)e_4.
\end{aligned}$$

In general, we let $\xi_t \in \mathbb{R}^2$ be the coordinates of $\bar{x}_t$ along $\mathrm{Span}\{e_{n-1}, e_n\}$, ($n$ being the index of the last column of a block relative to $\sigma \pm \mathrm{i}\omega$), and we let the same component of $\bar{u}_t$ be $\upsilon_t \in \mathbb{R}^2$, and of $\bar{w}_t$ be $\nu_t \in \mathbb{R}^2$. We have $\|\upsilon_t\| \leqslant \rho$ and $\mathsf{E}[\|\nu_t\|] \leqslant c_1$ for positive constants ρ, and c_1 as before. The dynamics of ξ_t is given by

$$\xi_{t+1} = \Lambda\xi_t + \upsilon_t + \nu_t,$$

where $\Lambda = \begin{bmatrix}\sigma & -\omega\\ \omega & \sigma\end{bmatrix}$. Define $\alpha = \sqrt{\sigma^2 + \omega^2}$, then $\|\Lambda\xi_t\| = \alpha\|\xi_t\|$. With essentially the same computations as before, we obtain

$$\begin{aligned}
&\mathsf{E}\left[\|\xi_{t+\tau}\| \mid \xi_\tau\right]\\
&\quad = \mathsf{E}\left[\left\|\Lambda^t\xi_\tau + \sum_{k=0}^{t-1}\Lambda^{t-1-k}\upsilon_{\tau+k} + \sum_{k=0}^{t-1}\Lambda^{t-1-k}\nu_{\tau+k}\right\| \,\middle|\, \xi_\tau\right]\\
&\quad \geqslant \alpha^t\|\xi_\tau\| - (\rho + c_1)\sum_{k=0}^{t-1}\alpha^k \geqslant \alpha^t\left(\|\xi_\tau\| - \beta\right).
\end{aligned}$$

Therefore, in both cases, if for a given $\varepsilon > 0$ we have $\|\xi_\tau\| \geqslant \beta + \varepsilon$, then $\mathsf{E}\big[\|\xi_{t+\tau}\| \,\big|\, \xi_\tau\big] \geqslant \alpha^t\varepsilon$, for a certain $\alpha > 1$. Let us now define the events

$$\begin{aligned}
H_1 &= \{\exists\tau \in \mathbb{N}_0 \mid \|\nu_\tau\| > (\alpha + 1)\beta + \rho\},\\
H_2 &= \{\exists\tau \in \mathbb{N}_0, \varepsilon > 0 \mid \|\xi_\tau\| \geqslant \beta + \varepsilon\}.
\end{aligned}$$

Note that $H_1 \subset H_2$. Indeed, if $\|\xi_t\| \leqslant \beta$ for all $t = 0, \ldots, \tau$ but $\|\nu_\tau\| > (\alpha + 1)\beta + \rho$, then

$$\|\xi_{\tau+1}\| \geqslant \|\nu_\tau\| - \|\Lambda\xi_\tau\| - \|\upsilon_\tau\| \geqslant (\alpha + 1)\beta + \rho - \alpha\beta - \rho > \beta.$$

By hypothesis, since ν_τ are unbounded, the random variables $1_{\{\|\nu_\tau\| > (\alpha+1)\beta+\rho\}}$ form a sequence of Bernoulli trials with a certain positive probability, hence $\mathsf{P}(H_1) = 1$. Consequently also $\mathsf{P}(H_2) = 1$, and

$$\begin{aligned}
\sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\|\bar{x}_t\|\big] &\geqslant \sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\|\xi_t\|\big] = \sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\|\xi_t\| \,\big|\, H_2\big]\\
&\geqslant \sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\|\xi_{t+\tau}\| \,\big|\, H_2\big]\\
&= \sup_{t\in\mathbb{N}_0}\mathsf{E}\Big[\mathsf{E}\big[\|\xi_{t+\tau}\| \,\big|\, \xi_\tau\big] \,\Big|\, H_2\Big] \geqslant \sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\alpha^t\varepsilon \,\big|\, H_2\big]\\
&= \sup_{t\in\mathbb{N}_0}\alpha^t\varepsilon\\
&= \infty.
\end{aligned}$$

Finally, $x_t = T\bar{x}_t$, where T represents the change of basis that brings A in Jordan form. In particular T is invertible, hence $\|\bar{x}_t\| \leqslant \|T^{-1}\|\|x_t\|$ and

$$\sup_{t\in\mathbb{N}_0}\mathsf{E}\big[\|x_t\|^2\big] \geqslant \sup_{t\in\mathbb{N}_0}\mathsf{E}[\|x_t\|]^2 \geqslant \frac{1}{\|T^{-1}\|^2}\sup_{t\in\mathbb{N}_0}\mathsf{E}[\|\bar{x}_t\|]^2 = \infty,$$

where the first inequality follows from Jensen's inequality. □

3. Simulations

Fig. 1 shows the average of the square norm of the state over 50 runs of the following system:

$$x_{t+1} = \begin{bmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{bmatrix}x_t + \begin{bmatrix}1\\0\end{bmatrix}u_t + w_t$$

where $x_0 = \begin{bmatrix}10\\10\end{bmatrix}$, $w_t \in \mathcal{N}(0, 0.1I_2)$, and where u_t is chosen respectively according to the policy proposed in this paper, the one proposed in [4], the one proposed in [7], and a slightly modified version of the policy in [7]. The policy proposed in [7] is

$$u_t = \sigma(-\rho B^\top Ax_t) \tag{3.1}$$

where σ is the component-wise saturation function, and ρ is a *small enough* constant. The modified version of the policy (3.1) is with a fixed $\rho = 1$. Although *no stability guarantee is provable* in this case, it is apparent that the policy (3.1) with $\rho = 1$ outperforms the other three choices in closed-loop. This fact supports our conjecture that mean-square stability is attainable also with a feedback policy.

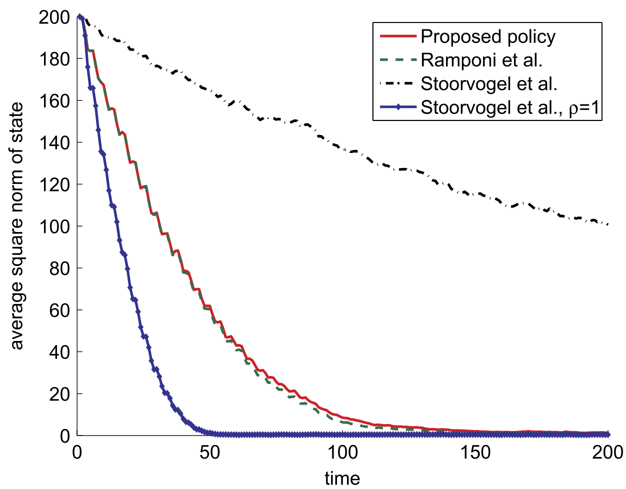

Fig. 1. Comparison of average squared norm of state: A orthogonal.

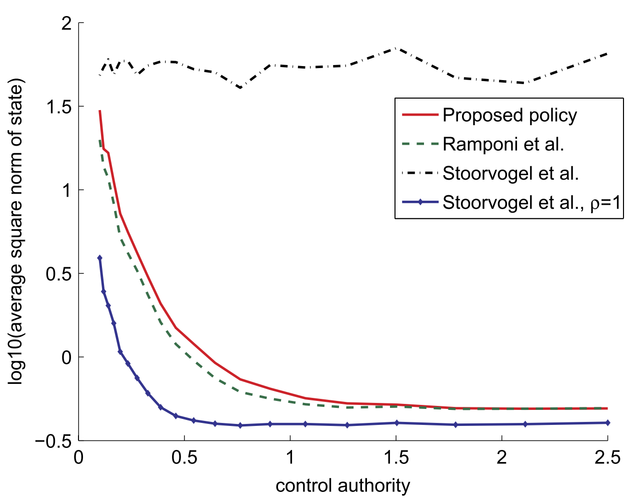

Fig. 2. Squared norm of state vs. bound R on u_t: A orthogonal.

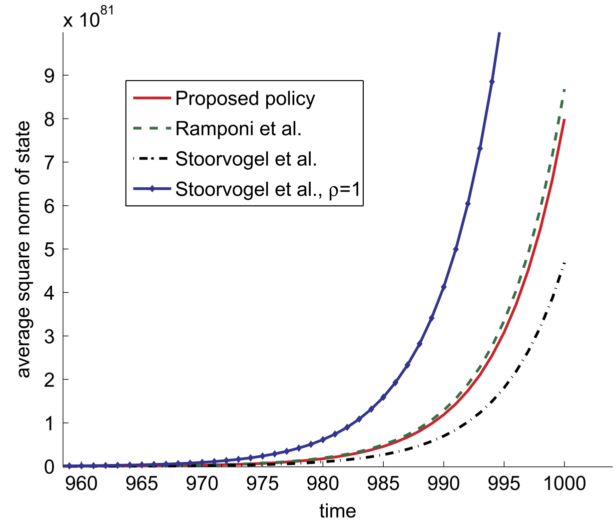

Fig. 3. Comparison of average squared norm of state: A unstable.

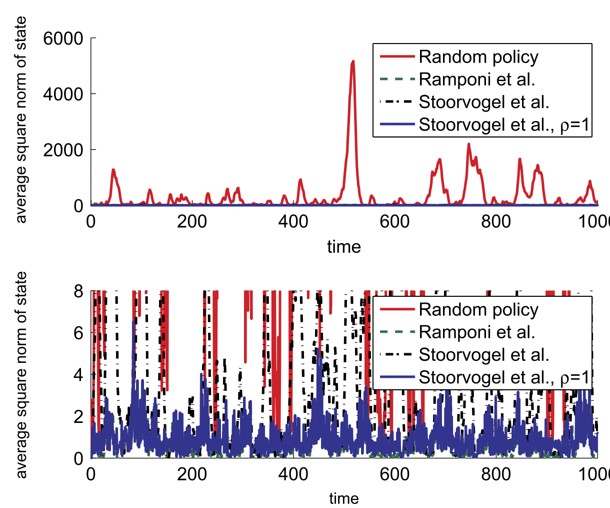

Fig. 4. Comparison of average squared norm of state: A asymptotically stable.

Fig. 2 shows the logarithm in base 10 of the average over 20 runs and 1000 time steps of the steady-state square norm, as a function of the control bound R, for the same four policies, now with initial condition $x_0 = 0$.

Fig. 3 shows the average of the square norm of the state over 50 runs of the following system, with an unstable A matrix, starting from zero initial condition:

$$x_{t+1} = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1.1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_t + w_t$$

where w_t is a white Gaussian noise with variance $0.2I$, respectively controlled with the policies proposed in this paper, the one proposed in [4], and the one proposed in [7]. Since the system has an eigenvalue outside the unit circle, sooner or later each realization "explodes" with exponential rate. This behavior illustrates Theorem 1.7.

Fig. 4 shows, on two different scales, the average of the square norm of the state over 50 runs of the following asymptotically stable system starting from zero initial condition:

$$x_{t+1} = \begin{bmatrix} 0.9 & 1 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.95 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_t + w_t$$

where w_t is a white Gaussian noise with variance $0.2I$, respectively controlled with a random policy, namely a Gaussian white noise with mean zero and unit variance, with the policy proposed in [4], and the one proposed in [7]. Since the system is asymptotically stable, the variance of the state remains bounded even with a random input.

Finally, Fig. 5 shows the average of the square norm of the state over 50 runs of the following system starting from zero initial condition:

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + w_t$$

where w_t is a white Gaussian noise with variance $0.2I$, respectively controlled with the policy proposed in [4] and the one proposed in [11]. The latter has been tuned as follows:

$$u_t = f(x_t) = \begin{cases} -1 & \text{if } x_t^{(1)} > 10,\ x_t^{(2)} \geqslant -10, \\ 1 & \text{if } x_t^{(1)} > 10,\ x_t^{(2)} \leqslant 10, \\ 0 & \text{otherwise.} \end{cases}$$

Both policies seem to attain mean square boundedness, although in neither case this has been proven. The behavior of the other policies considered earlier (namely the one proposed in [7]) is not

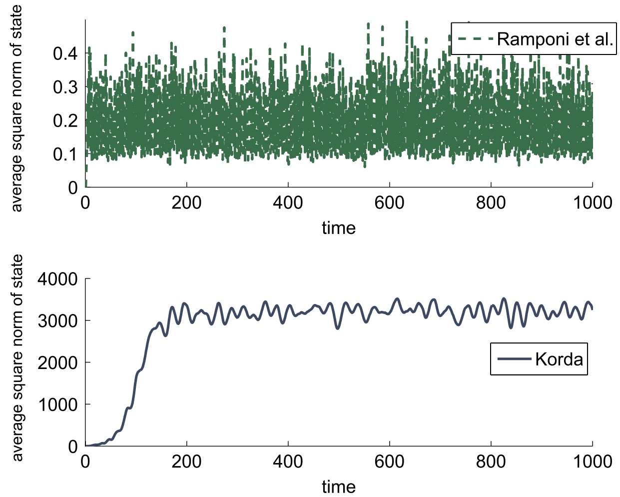

Fig. 5. Average squared norm of state: A having two eigenvalues equal to 1 and a non-trivial Jordan block.

clear at all. Note that for this system the policy proposed exhibits a stable behavior, despite the fact that powers of the A matrix are unbounded.

4. Conclusions

We established answers to the problem of attaining mean-square bounds for controlled linear systems with additive stochastic noise in two special cases:

- When the system matrix A is marginally stable and the controlled system is reachable, a bounded state feedback policy with finite memory exists to render the system mean square bounded starting from any initial state.
- When the matrix A has a spectral radius larger than one and the noise is unbounded in probability along the directions of unstable eigen-subspace of A, the system cannot be made mean square bounded by any bounded control policy.

Acknowledgments

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References

[1] H.J. Sussmann, E.D. Sontag, Y. Yang, A general result on the stabilization of linear systems using bounded controls, IEEE Transactions on Automatic Control 39 (12) (1994) 2411–2425.
[2] Y.D. Yang, E.D. Sontag, H.J. Sussmann, Global stabilization of linear discrete-time systems with bounded feedback, Systems and Control Letters 30 (5) (1997) 273–281.
[3] S.P. Meyn, R.L. Tweedie, Markov Chains and Stochastic Stability, second ed., Cambridge University Press, Cambridge, 2009.
[4] F. Ramponi, D. Chatterjee, A. Milias-Argeitis, P. Hokayem, J. Lygeros, Attaining mean square boundedness of a marginally stable stochastic linear system with a bounded control input, IEEE Transactions on Automatic Control 55 (10) (2010) 2414–2418.
[5] P. Hokayem, E. Cinquemani, D. Chatterjee, F. Ramponi, J. Lygeros, Stochastic receding horizon control with output feedback and bounded control inputs, Automatica 48 (1) (2012) 77–88.
[6] D. Chatterjee, S. Amin, P. Hokayem, J. Lygeros, S. Shankar Sastry, 2010, Mean-square boundedness of stochastic networked control systems with bounded control inputs, in: Proceedings of the 49th IEEE Conference on Decision and Control, pp. 4759–4764, http://arxiv.org/abs/1004.0793.
[7] A.A. Stoorvogel, A. Saberi, S. Weiland, 2007, On external semi-global stochastic stabilization of linear systems with input saturation, in: Proceedings of the American Control Conference, pp. 5845–5850.
[8] G.N. Nair, R.J. Evans, Stabilizability of stochastic linear systems with finite feedback data rates, SIAM Journal on Control and Optimization 43 (2) (2004) 413–436. electronic.
[9] S. Yüksel, T. Başar, 2010, Control over noisy forward and reverse channels, (in press) in IEEE Transactions on Automatic Control, vol. 56, Also available at http://www.mast.queensu.ca/~yuksel/YukBasNoisyTAC2011.pdf.
[10] R. Pemantle, J.S. Rosenthal, Moment conditions for a sequence with negative drift to be uniformly bounded in L^r, Stochastic Processes and their Applications 82 (1) (1999) 143–155.
[11] M. Korda, 2011, Stochastic Model: Predictive Control, Master thesis, ETH Zürich.